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Electromagnetic wave scattering in an imperfectly conducting open-ended waveguide

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The Wiener-Hopf method is used to obtain an exact solution to the problem of diffraction by imperfectly conducting parallel plates. The source point is assumed to be far from the waveguide so that the incident spherical wave is locally plane. A comparison has been made with the case of a perfectly conducting parallel plates waveguide.

1. Introduction

Scattering from a waveguide is a well-studied problem in diffraction theory. The names Schwinger, Heins, Carlson come to mind, and most of the results can be found in [1]. Related studies are in [2]–[8]. Only in a very limited number of cases have exact solutions of diffraction problems been obtained, and in all of them it has been assumed that the diffracting structures are of infinite conductivity. One such problem which is amenable to treatment is that of a perfectly conducting wedge, with the half-plane as a special case, based on which the pioneer work was done by Poincare [9], [10]. He succeeded in deriving the correct asymptotic field for a wedge, but it is Sommerfeld [11] to whom the credit is due for the first exact solution of diffraction at a plate. Later on the diffraction of electromagnetic waves from perfectly conducting obstacles, on which the tangential component of the electric field vanishes, has been treated by many authors. In practice, however, obstacles which have perfect conductivity are unlikely to be encountered and, therefore, it seems appropriate to investigate whether solutions are obtainable when better approximations to the boundary conditions are used in formulating a particular diffraction problem. In view of the considerable amount of information available about diffraction it seems useful to examine the diffraction problems having impedance boundary conditions. Raman and Krishnan [12] have treated both the half-plane and wedge of finite conductivity using a modified Sommerfeld's solution in which the image wave is multiplied by the Fresnel reflection coefficient for the screen; but their method appears somewhat artificial and violates the reciprocity condition concerning the interchangeability of transmitter and receiver. For a metallic wedge, Jones and Pidduck [13] have determined the diffracted wave at large angles. Subsequently, employing these impedance boundary conditions, solutions have been obtained by a half-plane or metallic sheet by Senior [14], [15] and

Williams [16], and for the problem of diffraction by a wedge, by Williams [17], [18] and Senior [19]. Faulkner [20] extended this analysis to the problem of plane electromagnetic wave by a metallic strip.

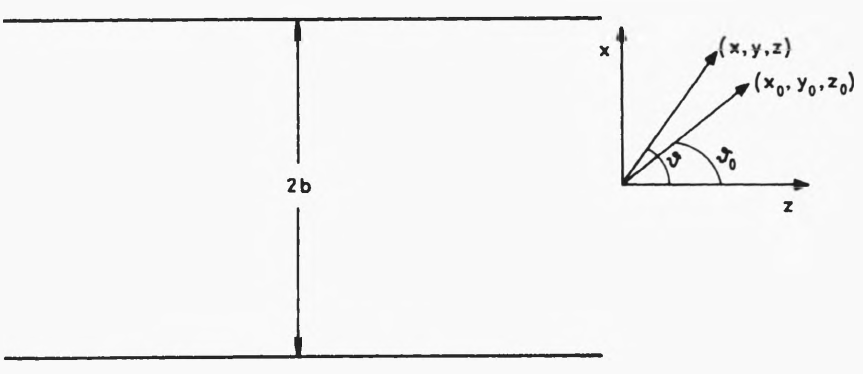
However, no attempt has been made to discuss the diffraction of a spherical wave in an open-ended waveguide satisfying the impedance boundary conditions. Impedance boundary conditions give rise to new mathematical complications. The essential part of this paper includes the following sections. Section 2 is devoted to formulation of the problem. In Section 3, the problem is solved by means of standard Wiener – Hopf technique [21] and an exact solution is obtained for the diffracted field. In Section 4, the saddle point method [22] and Cauchy's residue theorem are used to solve the integrals appearing in the inverse Fourier transform. Finally, concluding remarks are given in Section 5.

2. Formulation of the problem

Let (x, y, z) be rectangular Cartesian coordinates. Then, consider a parallel-plate waveguide with imperfectly conducting plates at $x = \pm b$. The geometry of the problem is shown in the Figure. The time harmonic factor $e^{-i\omega t}$ (ω is the angular frequency) is assumed and will be suppressed throughout. We consider a point source of unit strength to be located at (x_0, y_0, z_0). The total field $H_y^t(x, y, z)$ then satisfies the inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)H_y^t = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \tag{1}$$

where $k = k_1 + ik_2$.



Open-ended parallel plates waveguide

The impedance boundary conditions at $x = \pm b$ are given by

$$H_y^t(x, y, z) = \pm i\delta \frac{\partial}{\partial x} H_y^t(x, y, z), \quad -\infty < z < 0, \quad -\infty < y < \infty \tag{2}$$

where: $\delta = \frac{1}{kn}$ (k is the wave number and n is the complex refractive index of the plates). We remark that $\delta = 0$ corresponds to the boundary condition $H_y^t(x, y, z) = 0$ and $\delta = \infty$ corresponds to the boundary condition $\frac{\partial}{\partial x} H_y^t(x, y, z) = 0$. These are the usual boundary conditions for the insulating and perfectly conducting plates. The $\pm$ signs in Eq. (2) correspond to the upper and lower sides of each plate, respectively.

It is convenient to write the total field as

$$H_y^t(x, y, z) = H_y^i(x, y, z) + H_y^{sca}(x, y, z) \tag{3}$$

where H_y^i is the solution of inhomogeneous wave equation (1) that corresponds to the incident wave and H_y^{sca} is the solution of homogeneous version of wave equation (1) that corresponds to the scattered field. Thus, H_y^i and H_y^{sca} satisfy the following equations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_y^i = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \tag{4}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_y^{sca} = 0 \tag{5}$$

where Eq. (4) is satisfied at the point (x_0, y_0, z_0) and Eq. (5) is satisfied everywhere in space except at (x_0, y_0, z_0).

In addition, we insist that H_y^{sca} represents an outward radiating wavefield and satisfies the edge conditions [22]

$$\left.\begin{aligned} & H_y^t(x, y, z) = O(1), \\ & \frac{\partial}{\partial x} H_y^t(x, y, z) = O(r^{-1/2}) \quad \text{as } r \to 0 \end{aligned}\right\} \tag{5a}$$

where r is the distance from (b, y, z) to $(b, y, 0)$ or from $(-b, y, z)$ to $(-b, y, 0)$, respectively, with $z > 0$.

3. Solution of the problem

The Fourier transform and its inverse over the variable y are defined as

$$\left.\begin{aligned} & \varphi_t(x, \zeta, z) = \int_{-\infty}^{\infty} H_y^t(x, y, z) e^{-ik\zeta y} dy, \\ & H_y^t(x, y, z) = \frac{k}{2\pi} \int_{-\infty}^{\infty} \varphi_i(x, \zeta, z) e^{ik\zeta y} d\zeta. \end{aligned}\right\} \tag{6}$$

The transform parameter is taken as $k\zeta$ and ζ is non-dimensional. For analytic convenience, k is assumed to be complex and has a small positive imaginary part. The decomposition (6) is common in other field theories as well, for example, Fourier optics [23], [24]. Using Equation (6), the problem becomes:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k^2\lambda^2\right]\varphi_i(x,\zeta,z)=e^{-ik\zeta y_0}\delta(x-x_0)\delta(z-z_0), \tag{7}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k^2\lambda^2\right]\varphi(x,\zeta,z)=0, \tag{8}$$

$$\varphi_t(x,\zeta,z)=\pm i\delta\frac{\partial}{\partial x}\varphi_t(x,\zeta,z),\quad -\infty<z<0, \tag{9}$$

$$\varphi_t(x,\zeta,z)=\Phi_i(x,\zeta,z)+\varphi(x\cdot\zeta,z), \tag{10}$$

where $\lambda^2=(1-\zeta^2)$.

If $\psi(x,\zeta,\alpha)$ is the Fourier transform with respect to z of $\varphi(x,\zeta,z)$, *i.e.*,

$$\psi(x,\zeta,\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x,\zeta,z)e^{i\alpha z}dz=\psi_+(x,\zeta,\alpha)+\psi_-(x,\zeta,\alpha) \tag{11}$$

where:

$$\psi_+(x,\zeta,\alpha)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\varphi(x,\zeta,z)e^{i\alpha z}dz,$$

$$\psi_-(x,\zeta,\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\varphi(x,\zeta,z)e^{i\alpha z}dz,$$

$\alpha=\sigma+i\tau$ and the inverse transform which lies along the real line is

$$\varphi(x,\zeta,z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\psi_+(x,\zeta,\alpha)+\psi_-(x,\zeta,\alpha)]e^{-i\alpha z}d\alpha. \tag{12}$$

The solution of inhomogeneous wave equation (7) can be written as

$$\varphi_i=-\frac{e^{-ik\zeta y_0}}{4i(2\pi)^{1/2}}H_0^{(1)}(k\lambda[(x-x_0)^2+(z-z_0)^2]^{1/2})=E(\zeta)e^{-ik\lambda(x\sin\vartheta_0+z\cos\vartheta_0)} \tag{13}$$

where

$$E(\zeta)=i\frac{e^{-ik\zeta y_0}}{4\pi(k\lambda r_0)^{1/2}}e^{i(k\lambda r_0-\pi/4)}, \tag{14}$$

$$r_0^2=(x_0^2+z_0^2),\quad r_0\to\infty\ \text{and}\ 0<\vartheta_0<\pi/2.$$

Note that in Equation (11), $\psi_+(x,\zeta,\alpha)$ is regular (for $\mathrm{Im}\alpha>-\mathrm{Im}k\lambda$) in the upper half of the complex α-plane, $\psi_-(x,\zeta,\alpha)$ is regular (for $\mathrm{Im}\alpha<\mathrm{Im}k\lambda\cos\vartheta_0$)) in the lower half-plane.

Application of a Fourier transform with respect to z to Eq. (8) leads to

$$\psi(x,\zeta,\alpha)=\begin{cases}A(\alpha)e^{-\gamma x}, & x>b,\\ B(\alpha)e^{-\gamma x}+C(\alpha)e^{\gamma x}, & -b<x<b,\\ D(\alpha)e^{\gamma x}, & x<-b\end{cases} \tag{15}$$

where $\gamma = \sqrt{(\alpha^2 - k^2\lambda^2)}$, that branch of γ being chosen such that $\gamma = -ik\lambda$ when $\alpha = 0$.

It may be deduced from Equations (15) that

$$\psi_+(b,\zeta,\alpha)+\psi_-(b^+,\zeta,\alpha) = A(\alpha)e^{-\gamma b}, \tag{16a}$$

$$\psi_+(b,\zeta,\alpha)+\psi_-(b^-,\zeta,\alpha) = B(\alpha)e^{-\gamma b}+C(\alpha)e^{\gamma b}, \tag{16b}$$

$$\psi_+(-b,\zeta,\alpha)+\psi_-(-b^+,\zeta,\alpha) = B(\alpha)e^{\gamma b}+C(\alpha)e^{-\gamma b}, \tag{16c}$$

$$\psi_+(-b,\zeta,\alpha)+\psi_-(-b^-,\zeta,\alpha) = D(\alpha)e^{-\gamma b}, \tag{16d}$$

$$\psi'_+(b,\zeta,\alpha)+\psi'_-(b^+,\zeta,\alpha) = -\gamma A(\alpha)e^{-\gamma b}, \tag{17a}$$

$$\psi'_+(b,\zeta,\alpha)+\psi'_-(b^-,\zeta,\alpha) = -\gamma B(\alpha)e^{-\gamma b}+\gamma C(\alpha)e^{\gamma b}, \tag{17b}$$

$$\psi'_+(-b,\zeta,\alpha)+\psi'_-(-b^+,\zeta,\alpha) = -\gamma B(\alpha)e^{\gamma b}+\gamma C(\alpha)e^{-\gamma b}, \tag{17c}$$

$$\psi'_+(-b,\zeta,\alpha)+\psi'_-(-b^-,\zeta,\alpha) = \gamma D(\alpha)e^{-\gamma b} \tag{17d}$$

where

$$\psi_+(\pm b^+,\zeta,\alpha) = \psi_+(\pm b^-,\zeta,\alpha) = \psi_+(\pm b,\zeta,\alpha),$$

and primes indicate differentiation with respect to x.

Transforming the boundary conditions (9) we have

$$\psi_-(b^+,\zeta,\alpha) = i\delta\psi'_-(b^+,\zeta,\alpha)-\frac{E(\zeta)}{(2\pi)^{1/2}i}\left(\frac{1-k\lambda\delta\sin\vartheta_0}{\alpha-k\lambda\cos\vartheta_0}\right)e^{-ik\lambda b\sin\vartheta_0}, \tag{18a}$$

$$\psi_-(b^-,\zeta,\alpha) = -i\delta\psi'_-(b^-,\zeta,\alpha)-\frac{E(\zeta)}{(2\pi)^{1/2}i}\left(\frac{1+k\lambda\delta\sin\vartheta_0}{\alpha-k\lambda\cos\vartheta_0}\right)e^{-ik\lambda b\sin\vartheta_0}, \tag{18b}$$

$$\psi_-(-b^+,\zeta,\alpha) = i\delta\psi'_-(-b^+,\zeta,\alpha)-\frac{E(\zeta)}{(2\pi)^{1/2}i}\left(\frac{1-k\lambda\delta\sin\vartheta_0}{\alpha-k\lambda\cos\vartheta_0}\right)e^{ik\lambda b\sin\vartheta_0}, \tag{18c}$$

$$\psi_-(-b^-,\zeta,\alpha) = -i\delta\psi'_-(-b^-,\zeta,\alpha)-\frac{E(\zeta)}{(2\pi)^{1/2}i}\left(\frac{1+k\lambda\delta\sin\vartheta_0}{\alpha-k\lambda\cos\vartheta_0}\right)e^{ik\lambda b\sin\vartheta_0} \tag{18d}$$

where: b^+ and b^- denote the upper and lower sides of the plates at $x = \pm b$.

From Equations (16) and (17) we have:

$$A(\alpha) = \left[J_-(b,\zeta,\alpha)-\frac{1}{\gamma}J'_-(b,\zeta,\alpha)\right]e^{\gamma b}-\left[J_-(-b,\zeta,\alpha)-\frac{1}{\gamma}J'_-(-b,\zeta,\alpha)\right]e^{-\gamma b}, \tag{19a}$$

$$B(\alpha) = -\left[J_-(-b,\zeta,\alpha)-\frac{1}{\gamma}J'_-(-b,\zeta,\alpha)\right]e^{-\gamma b}, \tag{19b}$$

$$C(\alpha) = -\left[J_-(b,\zeta,\alpha)+\frac{1}{\gamma}J'_-(b,\zeta,\alpha)\right]e^{-\gamma b}, \tag{19c}$$

$$D(\alpha) = \left[J_-(-b,\zeta,\alpha)+\frac{1}{\gamma}J'_-(b,\zeta,\alpha)\right]e^{\gamma b}-\left[J_-(b,\zeta,\alpha)+\frac{1}{\gamma}J'_-(b,\zeta,\alpha)\right]e^{-\gamma b}, \tag{19d}$$

where:

$$\left.\begin{aligned}
J_-(b,\zeta,\alpha) &= \frac{1}{2}[\psi_-(b^+,\zeta,\alpha)-\psi_-(b^-,\zeta,\alpha)],\\
J_-(-b,\zeta,\alpha) &= \frac{1}{2}[\psi_-(-b^-,\zeta,\alpha)-\psi_-(-b^+,\zeta,\alpha)],\\
J'_-(b,\zeta,\alpha) &= \frac{1}{2}[\psi'_-(b^+,\zeta,\alpha)-\psi'_-(b^-,\zeta,\alpha)],\\
J'_-(-b,\zeta,\alpha) &= \frac{1}{2}[\psi'_-(-b^-,\zeta,\alpha)-\psi'_-(-b^+,\zeta,\alpha)].
\end{aligned}\right\} \tag{20}$$

Now, from Equations (17)–(19) we obtain

$$\psi'_+(b,\zeta,\alpha)+J_-(b,\zeta,\alpha)\left(\frac{1}{i\delta}+\gamma\right)+\frac{E(\zeta)k\lambda\sin\vartheta_0}{(2\pi)^{1/2}(\alpha-k\lambda\cos\vartheta_0)}e^{-ik\lambda b\sin\vartheta_0}$$

$$=\gamma e^{-2\gamma b}\left[J_-(-b,\zeta,\alpha)-\frac{1}{\gamma}J'_-(-b,\zeta,\alpha)\right], \tag{21}$$

$$\psi'_+(-b,\zeta,\alpha)-J_-(b,\zeta,\alpha)\left(\frac{1}{i\delta}+\gamma\right)+\frac{E(\zeta)k\lambda\sin\vartheta_0}{(2\pi)^{1/2}(\alpha-k\lambda\cos\vartheta_0)}e^{ik\lambda b\sin\vartheta_0}$$

$$=-\gamma e^{-2\gamma b}\left[J_-(b,\zeta,\alpha)+\frac{1}{\gamma}J'_-(b,\zeta,\alpha)\right], \tag{22}$$

$$\psi_+(b,\zeta,\alpha)+J'_-(b,\zeta,\alpha)\left(\frac{1}{\gamma}+i\delta\right)-\frac{E(\zeta)e^{ik\lambda b\sin\vartheta_0}}{i(2\pi)^{1/2}(\alpha-k\lambda\cos\vartheta_0)}$$

$$=-e^{-2\gamma b}\left[J_-(-b,\zeta,\alpha)-\frac{1}{\gamma}J'_-(-b,\zeta,\alpha)\right], \tag{23}$$

$$\psi_+(-b,\zeta,\alpha)-J'_-(-b,\zeta,\alpha)\left(\frac{1}{\gamma}+i\delta\right)-\frac{E(\zeta)e^{-ik\lambda b\sin\vartheta_0}}{i(2\pi)^{1/2}(\alpha-k\lambda\cos\vartheta_0)}$$

$$=-e^{-2\gamma b}\left[J_-(b,\zeta,\alpha)-\frac{1}{\gamma}J'_-(b,\zeta,\alpha)\right]. \tag{24}$$

Next, adding and subtracting Equations (21) and (22), (23) and (24) and then using the extended form of Liouville's theorem in the resulting expressions we arrive at

$$S_-(\alpha)=\frac{\frac{1}{i\delta}U_-(\alpha)-V_-(\alpha)}{2b(\alpha-k\lambda)G_-(\alpha)}+\frac{E(\zeta)(\alpha-k\lambda)^{-1}ik\lambda\sin\vartheta_0\sin(k\lambda b\sin\vartheta_0)}{b(2\pi)^{1/2}(\alpha-k\lambda\cos\vartheta_0)(k\lambda+k\lambda\cos\vartheta_0)G_+(k\lambda\cos\vartheta_0)G_-(\alpha)}, \tag{25a}$$

$$D_-(\alpha) = -\frac{\frac{1}{i\delta}X_-(\alpha)+Y_-(\alpha)}{2(\alpha-k\lambda)^{1/2}L_-(\alpha)}$$

$$-\frac{E(\zeta)(\alpha-k\lambda)^{-1/2}k\lambda\sin\vartheta_0\cos(k\lambda b\sin\vartheta_0)}{(2\pi)^{1/2}(\alpha-k\lambda\cos\vartheta_0)(k\lambda+k\lambda\cos\vartheta_0)^{1/2}L_+(k\lambda\cos\vartheta_0)L_-(\alpha)}, \tag{25b}$$

$$T'_-(\alpha) = \frac{i\delta N_-(\alpha)-F_-(\alpha)}{2bG_-(\alpha)}+\frac{E(\zeta)\sin(k\lambda b\sin\vartheta_0)}{(2\pi)^{1/2}(\alpha-k\lambda\cos\vartheta_0)G_+(k\lambda\cos\vartheta_0)G_-(\alpha)}, \tag{25c}$$

$$R'_-(\alpha) = -\frac{i\delta W_-(\alpha)+I_-(\alpha)}{2(\alpha-k\lambda)^{-1/2}L_-(\alpha)}+\frac{E(\zeta)(\alpha-k\lambda)^{1/2}(\alpha+k\lambda)^{1/2}\cos(k\lambda b\sin\vartheta_0)}{(2\pi)^{1/2}i(\alpha-k\lambda\cos\vartheta_0)L_+(k\lambda\cos\vartheta_0)L_-(\alpha)}. \tag{25d}$$

In Equations (25),

$$\gamma = (\alpha^2-k^2\lambda^2)^{1/2} = (\alpha+k\lambda)^{1/2}(\alpha-k\lambda)^{1/2},$$

$$G(\alpha) = \frac{e^{-\gamma b}\sinh\gamma b}{\gamma b} = G_+(\alpha)G_-(\alpha) = G_+(\alpha)G_+(-\alpha),$$

$$L(\alpha) = e^{-\gamma b}\cosh\gamma b = L_+(\alpha)L_-(\alpha) = L_+(\alpha)L_+(-\alpha),$$

$$\frac{S_-(\alpha)}{(\alpha+k\lambda)G_+(\alpha)} = U_+(\alpha)+U_-(\alpha),$$

$$\frac{R'_-(\alpha)e^{-2\gamma b}}{(\alpha+k\lambda)G_+(\alpha)} = V_+(\alpha)+V_-(\alpha),$$

$$\frac{D_-(\alpha)}{(\alpha+k\lambda)^{1/2}L_+(\alpha)} = X_+(\alpha)+X_-(\alpha), \quad \frac{T'_-(\alpha)}{G_+(\alpha)} = N_+(\alpha)+N_-(\alpha),$$

$$\frac{T'_-(\alpha)e^{-2\gamma b}}{(\alpha+k\lambda)^{1/2}L_+(\alpha)} = Y_+(\alpha)+Y_-(\alpha), \quad \frac{D_-(\alpha)e^{-2\gamma b}}{G_+(\alpha)} = F_+(\alpha)+F_-(\alpha),$$

$$\frac{R'_-(\alpha)(\alpha+k\lambda)^{1/2}}{L_+(\alpha)} = W_+(\alpha)+W_-(\alpha),$$

$$\frac{S_-(\alpha)e^{-2\gamma b}(\alpha+k\lambda)^{1/2}}{L_+(\alpha)} = I_+(\alpha)+I_-(\alpha),$$

$$\left.\begin{aligned} S_-(\alpha) &= J_-(b,\zeta,\alpha)+J_-(-b,\zeta,\alpha),\\ D_-(\alpha) &= J_-(b,\zeta,\alpha)-J_-(-b,\zeta,\alpha),\\ R'_-(\alpha) &= J'_-(b,\zeta,\alpha)-J'_-(-b,\zeta,\alpha),\\ T'_-(\alpha) &= J'_-(b,\zeta,\alpha)+J'_-(-b,\zeta,\alpha), \end{aligned}\right\} \tag{26}$$

and by [25],

$$G_+(\alpha) = \left(\frac{\sin k\lambda b}{k\lambda b}\right)^{1/2} \exp\left\{\frac{ib\alpha}{\pi}[1 - C_1 + \ln(2\pi/k\lambda b) + i\pi/2]\right\}$$

$$\times \exp\left\{\frac{ib\gamma}{\pi}\ln((\alpha-\gamma)/k\lambda)\right\} \prod_{\substack{n=2\\ \text{even}}}^{\infty} (1 + \alpha/i\gamma_n)e^{i2\alpha b/n\pi},$$

$$L_+(\alpha) = (\cos k\lambda b)^{1/2} \exp\left\{\frac{ib\alpha}{\pi}[1 - C_1 + \ln(\pi/2k\lambda b) + i\pi/2]\right\}$$

$$\times \exp\left\{\frac{ib\gamma}{\pi}\ln((\alpha-\gamma)/k\lambda)\right\} \prod_{\substack{n=1\\ \text{even}}}^{\infty} (1 + \alpha/i\gamma_n)e^{i2\alpha b/n\pi},$$

$\gamma_n = [(n\pi/2b)^2 - k^2\lambda^2]^{1/2}$,

$C_1 = 0.57721$ ... is the Euler's constant.

Using Equations (19) and (26) we can write:

$$B(\alpha) + C(\alpha) = -\left[S_-(\alpha) + \frac{1}{\gamma}R'_-(\alpha)\right]e^{-\gamma b}, \tag{27}$$

$$B(\alpha) - C(\alpha) = \left[D_-(\alpha) + \frac{1}{\gamma}T'_-(\alpha)\right]e^{-\gamma b}. \tag{28}$$

Equations (12) and (15) together with Eqs. (27) and (28) yield

$$\varphi(x,\zeta,z) = -\frac{1}{(2\pi)^{1/2}} \int_{-\infty+i\tau}^{\infty+i\tau} \left[S_-(\alpha)\cosh\gamma x + D_-(\alpha)\sinh\gamma x\right.$$

$$\left. + \frac{1}{\gamma}R'_-(\alpha)\cosh\gamma x + \frac{1}{\gamma}T'_-(\alpha)\sinh\gamma x\right]e^{-\gamma b - i\alpha z}d\alpha, \tag{29}$$

where: $-\mathrm{Im}k\lambda < \mathrm{Im}\alpha < \mathrm{Im}k\lambda\cos\vartheta_0$.

4. Field within the waveguide

The transmitted field inside the waveguide can be calculated from Eq. (29). For negative z, we enclose the contour of integration in the upper half-plane. The integrand has simple poles at (i) $\alpha = k\lambda$ and $\alpha = k\lambda\cos\vartheta_0$, (ii) $\alpha = i\gamma_n$ $(n = 0, 2, 4, \ldots)$ in $S_-(\alpha)$ and $R'_-(\alpha)$ corresponding to the equation $G_-(\alpha) = 0$, (iii) $\alpha = i\gamma_n$ $(n = 1, 3, 5, \ldots)$ in $D_-(\alpha)$ and $T'_-(\alpha)$ corresponding to $L_-(\alpha) = 0$. Evaluating the residues, we have

$$\varphi(x,\zeta,z) = -E(\zeta)[e^{ik\lambda r\cos(\vartheta-\vartheta_0)} + e^{ik\lambda r\cos(\vartheta+\vartheta_0)}$$

$$-\mathscr{G}_1(k\lambda\cos\vartheta_0)e^{ik\lambda r\cos\vartheta} - \mathscr{G}_2(k\lambda\cos\vartheta_0) - \mathscr{G}_3(k\lambda)e^{ik\lambda r\cos\vartheta}] \tag{30}$$

where:

$$\mathcal{G}_1(k\lambda\cos\vartheta_0) = \frac{\sin(k\lambda b\sin\vartheta_0)}{k\lambda b\sin\vartheta_0 G_-(k\lambda)G_+(k\lambda\cos\vartheta_0)}, \tag{31a}$$

$$\mathcal{G}_2(k\lambda\cos\vartheta_0) = e^{\gamma_n r\cos\vartheta}\left\{\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{(-1)2k\lambda b\sin\vartheta_0\cos(k\lambda b\sin\vartheta_0)(k\lambda+i\gamma_n)^{1/2}}{n\pi(k\lambda+k\lambda\cos\vartheta_0)^{1/2}(i\gamma_n-k\lambda\cos\vartheta_0)L_+(k\lambda\cos\vartheta_0)L'_-(i\gamma_n)}\right.$$

$$+\sum_{\substack{n=2\\ \text{even}}}^{\infty}\frac{\sin\vartheta_0\sin(k\lambda b\sin\vartheta_0)}{b(1+\cos\vartheta_0)(k\lambda-i\gamma_n)(k\lambda\cos\vartheta_0-i\gamma_n)G_+(k\lambda\cos\vartheta_0)G'_-(i\gamma_n)}$$

$$+\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{\sin(k\lambda b\sin\vartheta_0)}{n\pi(k\lambda\cos\vartheta_0-i\gamma_n)G_+(k\lambda\cos\vartheta_0)G'_-(i\gamma_n)}$$

$$+\sum_{\substack{n=2\\ \text{even}}}^{\infty}\frac{\cos(k\lambda b\sin\vartheta_0)}{b(k\lambda\cos\vartheta_0-i\gamma_n)L_+(k\lambda\cos\vartheta_0)L'_-(i\gamma_n)}$$

$$\left.+[M_-(i\gamma_n)+Q_-(i\gamma_n)+H_-(i\gamma_n)+O_-(i\gamma_n)]\frac{1}{E(\zeta)}\right\}\cos(n\pi(x-b)/2b), \tag{31b}$$

$$\mathcal{G}_3(k\lambda) = -\frac{\frac{1}{i\delta}U_-(k\lambda)-V_-(k\lambda)}{2bG_-(k\lambda)E(\zeta)}, \tag{31c}$$

$$M_-(i\gamma_n) = -\frac{\frac{1}{i\delta}X_-(i\gamma_n)+Y_-(i\gamma_n)}{2(i\gamma_n-k\lambda)^{1/2}L'_-(i\gamma_n)},$$

$$Q_-(i\gamma_n) = -\frac{i\delta N_-(i\gamma_n)-F_-(i\gamma_n)}{2bG'_-(i\gamma_n)},$$

$$H_-(i\gamma_n) = -\frac{\frac{1}{i\delta}U_-(i\gamma_n)-V_-(i\gamma_n)}{2b(i\gamma_n-k\lambda)G'_-(i\gamma_n)},$$

$$O_-(i\gamma_n) = -\left\{\frac{i\delta W_-(i\gamma_n)+I_-(i\gamma_n)}{2L'_-(i\gamma_n)}\right\}(i\gamma_n-k\lambda)^{1/2},$$

$$L'_-(i\gamma_n) = \left.\frac{dL_-(\alpha)}{d\alpha}\right|_{\alpha=i\gamma_n}, \quad G'_-(i\gamma_n) = \left.\frac{dG_-(\alpha)}{d\alpha}\right|_{\alpha=i\gamma_n}.$$

Now, using Equations (14) and (30), in Equation (6) we have

$$H_y^{\text{sca}}(x,y,z) = \frac{e^{-3i\pi/4}}{8\pi^2}\left(\frac{k}{r_0}\right)^{1/2}\int_{-\infty}^{\infty}\frac{e^{ik[\lambda\{r\cos(\vartheta-\vartheta_0)+r_0\}]+\zeta(y-y_0)]}}{\sqrt{\lambda}}d\zeta$$

$$
-\frac{e^{-3i\pi/4}}{8\pi^2}\left(\frac{k}{r_0}\right)^{1/2}\int_{-\infty}^{\infty}\mathscr{G}_1(k\lambda\cos\vartheta_0)\frac{e^{ik[\lambda\{r\cos\vartheta+r_0\}+\zeta(y-y_0)]}}{\sqrt{\lambda}}d\zeta
$$

$$
-\frac{e^{-3i\pi/4}}{8\pi^2}\left(\frac{k}{r_0}\right)^{1/2}\int_{-\infty}^{\infty}\mathscr{G}_2(k\lambda\cos\vartheta_0)\frac{e^{ik[\lambda r_0+\zeta(y-y_0)]}}{\sqrt{\lambda}}d\zeta
$$

$$
-\frac{e^{-3i\pi/4}}{8\pi^2}\left(\frac{k}{r_0}\right)^{1/2}\int_{-\infty}^{\infty}\mathscr{G}_3(k\lambda)\frac{e^{ik[\lambda\{r_0+r\cos\vartheta\}+\zeta(y-y_0)]}}{\sqrt{\lambda}}d\zeta
$$

$$
+\frac{e^{-3i\pi/4}}{8\pi^2}\left(\frac{k}{r_0}\right)^{1/2}\int_{-\infty}^{\infty}\frac{e^{ik[\lambda\{r\cos(\vartheta+\vartheta_0)+r_0\}+\zeta(y-y_0)]}}{\sqrt{\lambda}}d\zeta. \tag{32}
$$

In order to solve the integrals appearing in Equations (32), we introduce the transformation $\zeta = \cos\xi(0 < \mathrm{Re}\,\xi < \pi)$. The integrals are then solved asymptotically by using saddle point method and the resulting expression is given by

$$
H_y^{sca}(x,y,z) = -\frac{e^{ikr_{11}}}{4\pi_{11}} + \frac{(\sin\vartheta_1)^{1/2}e^{ikr_1}}{4\pi(2\pi r_0 r_1)^{1/2}}\mathscr{G}_1(k\cos\vartheta_0\sin\vartheta_1)
+\frac{(\sin\vartheta_2)^{1/2}e^{ikr_2}}{4\pi(2\pi r_0 r_2)^{1/2}}\mathscr{G}_2(k\cos\vartheta_0\sin\vartheta_2)+\frac{(\sin\vartheta_1)^{1/2}e^{ikr_1}}{4\pi(2\pi r_0 r_1)^{1/2}}\mathscr{G}_3(k\sin\vartheta_1)-\frac{e^{ikr_{22}}}{4\pi r_{22}} \tag{33}
$$

where

$$
\begin{aligned}
r_{11}^2 &= (r_0+r\cos(\vartheta-\vartheta_0))^2+(y-y_0)^2,\\
r_1^2 &= (r_0+r\cos\vartheta)^2+(y-y_0)^2,\\
r_2^2 &= r_0^2+(y-y_0)^2,\\
r_{22}^2 &= (r_0+r\cos(\vartheta+\vartheta_0))^2+(y-y_0)^2,\\
kr_{11} &\to \infty, \quad kr_1 \to \infty, \quad kr_2 \to \infty, \quad kr_{22} \to \infty.
\end{aligned}
$$

5. Concluding remarks

A method based on the Wiener–Hopf technique has been presented which allows calculation of the electromagnetic field within imperfectly conducting parallel plates. This field shows good agreement with the results of MITTRA and LEE [25] when conductivity is infinite. Further, as a check if we allow $\delta = \infty$ expression (33) reduces to the known result [21] for the acoustic field of a spherical wave by rigid parallel plates. The present work is also of much use in acoustics because double plate systems have many important engineering applications, both in buildings and as components of aircraft and marine structures. Sometimes they are useful purpose of sound control or as a result of streamlining requirements. Such plates are used to connect each other mechanically, either by common frames used to stiffen the structure, or as a result of imperfect construction. Particularly, when the double plate structure is used for noise control it is important to understand the acoustic properties of such connections between the plates.

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