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Generalized Vander-Lugt filter*

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The effective pupil formalism is applied to analysis of modified holographic system in more general geometry than Vander–Lugt filter. The influence of both geometry of the system and film OTF on the resolving power is discussed.

Introduction

The influence of both geometry and film OTF on impulse response of holographic optical system (HOS) has been reported by several authors [1, 2], but the first general analysis of linear HOS in paraxial approximation, based on Kelly's model of the film [3], was given by Kozma and Zelenka [4] and then in [5], where effective pupil function has been introduced.

In this paper, the effective pupil formalism was applied to a more general holographic system which can be called: generalized Vander–Lugt filter system because, in its simpliest case, it is identical with conventional Vander–Lugt filter (see e.g. [6]). Such an analysis can be useful in data processing holographic devices, and also in some practical cases, for instance, when nonlinear effects in emulsion are met, the hologram plate must be slightly translated with respect to Fourier plane of the system [7], or when using of conventional lenses is not convenient. On the other hand, even in conventional Vander–Lugt filter case, the effective pupil formalism allows a more precise analysis, especially when the small apertures are considered.

Effective pupil function of HOS

In order to define effective pupil function of linear HOS, we shall present impulse response (or point-spread function) of the system, under assumption that the background terms and the image terms for an object are spatially separated. Then the background terms can be omitted. Using signification as in [5] we obtain impulse response function h in paraxial approximation (upper sign corresponds to primary image term and lower sign

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to conjugate one):

$$
\begin{aligned}
h(x,y;\, x_i, y_i) = {} & A\, \frac{\exp(jkz)}{j\lambda z} \exp\left[\left(\frac{jk}{2z}\right)(x^2+y^2)\right]\times \\
& \times \exp\left[\left(\frac{\pm jk}{2z_i}\right)(x_i^2+y_i^2)\right] \exp\left[\left(\frac{jk}{2z_0}\right)(x_0^2+y_0^2)\right]\times \\
& \times \iint\limits_{-\infty}^{+\infty} P_{\mathrm{eff}}(x_1, y_1;\, x_i, y_i) \exp\left[\frac{jk}{2}(x_1^2+y_1^2)\times \right. \\
& \left. \times\left(\frac{1}{z_0}+\frac{1}{z}\pm\frac{1}{z_i}\mp\frac{1}{z_r}\right)\right] \exp(-jk)\left\{\left[\frac{x_0}{z_0}+\frac{x}{z}\pm\frac{x_i+\left(\frac{z_i}{z_r}\right)s}{z_i}\right]x_1+\right. \\
& \left. +\left(\frac{y_0}{z_0}+\frac{y}{z}\pm\frac{y_i}{z_i}\right)y_1\right\} dx_1\, dy_1,
\end{aligned}
\tag{1}
$$

where λ is wavelength, A is a constant value and variables: x_i, y_i, z_i; s, z_r; x_0, y_0, z_0 determine geometry (fig. 1) of the object, reference and

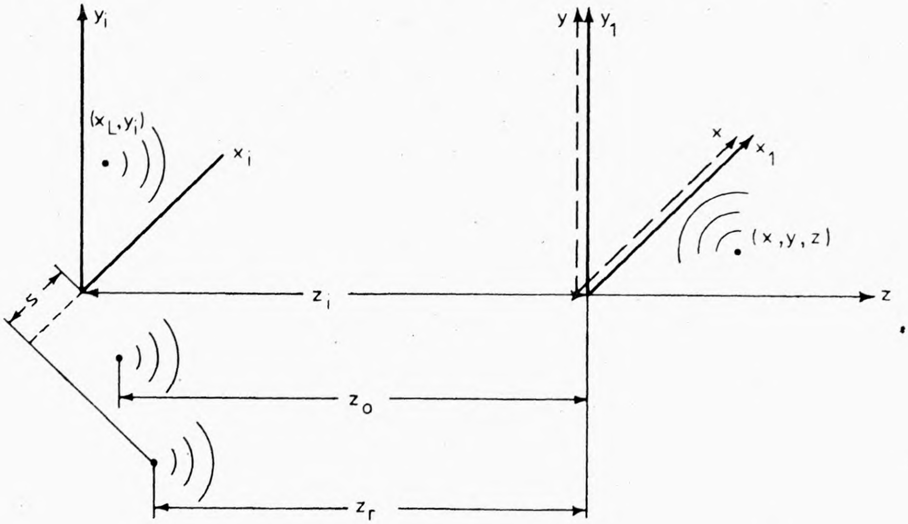

Fig. 1. Geometry of the HOS

reconstructing beams, respectively. Next, variables x_1, y_1; x, y, z are associated with the hologram plane and wavefront reconstruction space, respectively, and P_{eff} is effective pupil function of the HOS which has the following product form [5]:

$$
P_{\mathrm{eff}}(x_1, y_1;\, x_i, y_i) = P_F(x_1, y_1;\, x_i, y_i)P(x_1, y_1), \tag{2}
$$

where $P(x_1, y_1)$ is the conventional pupil function with $P(x_1, y_1) = 1$ inside the hologram plane and $P(x_1, y_1) = 0$ outside of the hologram, and P_F is a generalized pupil function for the film [5]:

$$P_F(x_1, y_1;\ x_i, y_i) = M\left\{\frac{|z_i - z_r|}{z_i z_r}[(x_1 - x_F)^2 + (y_1 - y_F)^2]^{1/2}\right\}. \tag{3}$$

The coordinates of the pupil centre in (x_1, y_1) plane have the form:

$$x_F = x_F(x_i) = \frac{\left[x_i + \left(\frac{z_i}{z_r}\right)s\right]z_r}{z_r - z_i},$$

$$y_F = y_F(y_i) = \frac{y_i z_r}{z_r - z_i}, \tag{4}$$

and $M(f)$ expressed as a function of local spatial frequency f is the Optical Transfer Function (OTF) with circular symmetry.

Consider the both particular cases interesting in practice: lensless Fourier HOS and lens Fourier HOS. For the first case we have $z_i = z_r$. Therefore from eq. (3) we find that the OTF aperturing effect occurs in object plane (x_i, y_i) instead of hologram plane (x_1, y_1) and the relation (3) is then replaced by an analogous one, with $x_F^{(1)} = -(s/z_r)z_i$ standing for x_F, and $y_F^{(1)} = 0$. For the second case the situation is similar, but formally we must exchange: $z_i \rightarrow F_i$ and $(s/z_i) \rightarrow \Theta_r$, where F_i – focal length of the lens in the recording process, and Θ_r – angle of reference plane wave; hence $x_F^{(2)} = -F_i\Theta_r$, and $y_F^{(2)} = 0$. Therefore, the generalized pupil function for these cases $(j = 1, 2)$ may be expressed in the form:

$$P_F^{(j)}(x_1, y_1;\ x_i, y_i) = P_F(x_i, y_i)$$
$$= M\left\{\frac{1}{\lambda z_i^{(j)}}[(x_i - x_F^{(j)})^2 + y_i^2]^{1/2}\right\}, \tag{5}$$

where: $z_j^{(1)} = z_i$, $z_i^{(2)} = F_i$.

For the typical model of OTF we have [8]:

$$M(f) = \exp\left[-\left(\frac{f}{f_0}\right)^n\right], \tag{6}$$

where $n \geqslant 1$. In particular, for $n \rightarrow \infty$, the OTF becomes jump function with spatial frequency cutoff f_0; hence, for this case, the formula (3) has the following form:

$$P_F^{(s)}(x_1, y_1;\ x_i, y_i) = \text{circus}\left\{\frac{[(x_1 - x_F)^2 + (y_1 - y_F)^2]^{1/2}}{R_F}\right\}, \tag{7}$$

where

$$R_F = \frac{\lambda z_i z_r f_0}{|z_i - z_r|}. \tag{8}$$

In fig. 2 we present geometrical relations between rectangular hologram plate and circles with radii R_F which characterize the pupil functions P_F for some discrete object points (x_{in}, y_{in}).

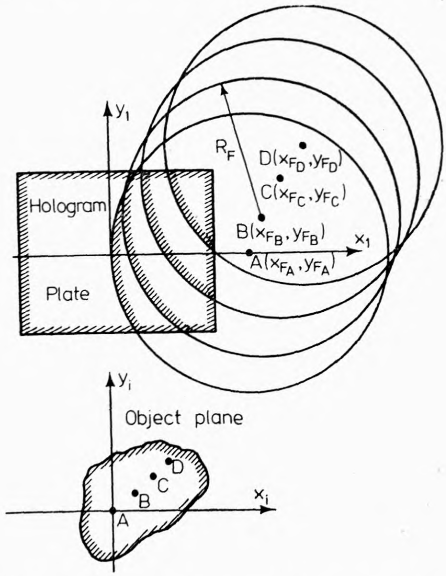

Fig. 2. Illustration of OTF influence, as a masking operation, for discrete point of the object

For the both Fourier HOS cases, according to eq. (5), the OTF pupil is situated in the object plane, and for $n = \infty$, we obtain formulae similar of eq. (7) but with $R_F^{(j)} = \lambda z_i^{(j)} f_0$.

It would be noted, that from eqs. (3)–(7) the HOS is not isoplanatic, in general, even in a paraxial approximation. Fortunatelly, for sufficiently small objects, the isoplanarity condition is usually fullfiled, although it depends on the sizes of the region Δf, where Modulation Transfer Function (MTF) is nearly constant. From the above OTF model, (see eq. (6)) we have $\Delta f = \eta^{(M)}(n)\cdot f_0$, where the shape coefficient η depends, of course, on the restriction required for ΔM in $(f, f+\Delta f)$ region. For example, when $1 \leqslant n \leqslant 2$, $\eta^{(0.1)} \approx 0.1$ and $\eta^{(0.01)} \approx 0.01$. Therefore, according to the above remarks, we get from eqs. (4), (5), (8) the following isoplanarity conditions:

$$\frac{|\Delta x_F|}{R_F} \leqslant \eta(n), \quad \text{for Fresnel HOS,} \tag{9a}$$

$$\frac{|\Delta x_i|}{R_F} \leqslant \eta(n), \quad \text{for Fourier HOS.} \tag{9b}$$

For optical frequencies and real materials, the basic parameter λf_0 (characterizing the film OTF influence on P_{eff}) is in range 0.1–0.2.

Assuming $f_0 = 0.1$, we obtain the following results, according to (9a, b): $|\Delta x_i|/z_i \leqslant 0.1\eta(n)$ for both Fresnel HOS and lensless Fourier HOS, and $|\Delta x_i|/F_i \leqslant 0.1\eta(n)$ for lens Fourier HOS.

For isoplanatic HOS, we can introduce coherent transfer function (CTF) H (see e.g. [6]) which in our case has the following form:

$$H(f_x, f_y) = P_{\text{eff}}(-\lambda z f_x, \ -\lambda z f_y; 0, 0) \times \exp\left\{ j2\pi f_x z \left(\frac{x_0}{z_0} \pm \frac{s}{z_r} \right) \right\}, \quad (10)$$

where f_x, f_y — spatial frequencies of the image.

The form of the CTF determines two-dimensional pass-band of the HOS and by the same means its resolving power. In the first approximation, the procedure for HOS resolution determination is as follows: for given hologram sizes we find $P(x, y)$ and for a given geometry of the system and cutoff frequency of the film f_0 we find $P_F^{(s)}$ (see eq. (7)) and, finally, $P_{\text{eff}}^{(s)}$. A simple example of such procedure is presented in fig. 3.

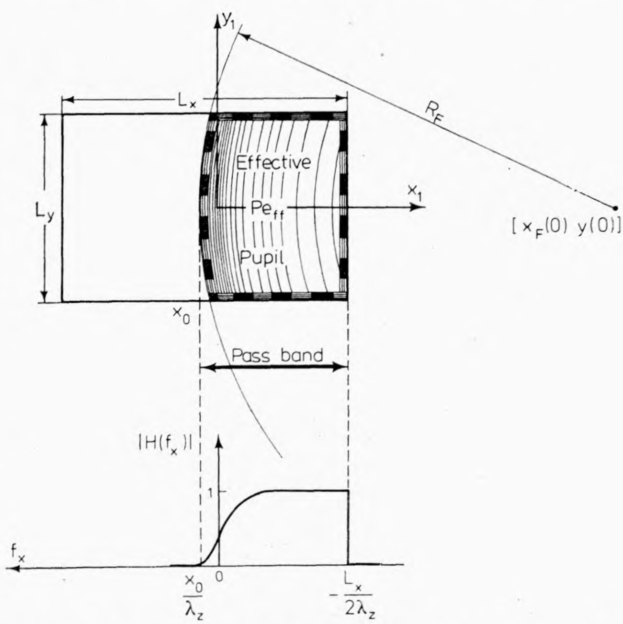

Fig. 3. Analogy between isoplanatic HOS and band-pass filter; transfer function H is shown for $f_y = 0$

We can see that the pass-band in x-direction is contained within the range: $L_x/2\lambda z \leqslant f_x \leqslant |x_0|/\lambda z$ and resolving power decreases about two times in comparison with the case when the OTF influence is omitted.

For general nonisoplanatic case, the above method can be easily generalized, because the eq. (10) holds also in isoplanatic region of arbitrary object; hence, when the object is locally isoplanatic, we get the following, general formula:

$$H(f_x, f_y;\ x_j\ , y_i) = P_{\text{eff}}(-\lambda z f_x, -\lambda z f_y;\ x_i y_i) \times \times \exp\left\{j 2\pi f_x z \left(\frac{x_0}{z_0} \pm \frac{s}{z_r} \pm \frac{x_i}{z_i}\right)\right\}. \quad (11)$$

The form of generalized CTF allows very interesting interpretation of the HOS resolution in information theory formalism: the pass-bands of the optical channel connected with HOS are different for several isoplanatic regions of the object.

Properties of generalized Vander-Lugt system

The HOS considered in previous chapter may be treated as a subsystem of a more general system, analysed below.

It would be noted that although the HOS is not generally isoplanatic in relation to the variables (x, y) and (x_i, y_i), it has this property in the case of the variables (x, y) and (x_0, y_0) (see eqs. (1), (2)). In other words, the isoplanatism of the system is assured by placing an object in the (x_0, y_0) plane, during the reconstructing process. Moreover, in this case, we have a complete analogy with conventional optical system. Really, from eq. (1), the image equation can be written in the form of Fresnel lens equation:

$$\frac{1}{z_0} + \frac{1}{z} = \frac{1}{F_H} \text{ with } \frac{1}{F_H} = \pm\left(\frac{1}{z_i} - \frac{1}{z_r}\right) \quad (12)$$

analogical to classical equation for the both lenses (for Fourier HOS we have $F_H = \infty$). By rescaling the x-variables in the following manner:

$$\tilde{x}_1 = \frac{x_1}{\lambda z}, \quad \tilde{x}_i = -m x_i, \quad \tilde{x}_0 = -m_0 x_0 \quad (13)$$

(the same being done for the y_1, y_i and y_0 variables), where $m = z/z_i$ — the magnification of HOS, and $m_0 = z/z_0$ — the magnification of Fresnel (holographic) lens, we obtain, for point in (x_i, y_i) plane, the following equation:

$$V_0(x, y) = \iint_{-\infty}^{+\infty} \tilde{h}^*(x - \tilde{x}_0, y - \tilde{y}_0)\, U_0(\tilde{x}_0, \tilde{y}_0)\, d\tilde{x}_0 d\tilde{y}_0, \quad (14)$$

where $\tilde{h}^*$ is the impulse response of the HOS (see eq. (1)) for conjugate image, after rescaling, $U_0(x_0, y_0)$ characterizes the object placed in the (x_0, y_0) plane. However the function $V_0(x, y)$ may be treated as a response of a more general holographic system called generalized Vander–Lugt filter on the point object situated in (x_i, y_i) plane.

Now, let us assume an isoplanatic HOS. Hence, according to conditions (9a, b) we get:

$$P_{\text{eff}}(x_1, y_1;\ x_i, y_i) = P_{\text{eff}}(x_1, y_1;\ 0, 0), \tag{15a}$$

and

$$\tilde{h}^*(x-\tilde{x}_0, y-\tilde{y}_0;\ \tilde{x}_i, \tilde{y}_i) = \tilde{h}^*(x-\tilde{x}_0-\tilde{x}_i;\ y-\tilde{y}_0-\tilde{y}_i). \tag{15b}$$

By placing an additional object in (x_i, y_i) plane (see fig. 4) and rescaling the variables, we obtain a double convolution relation in the conjugate

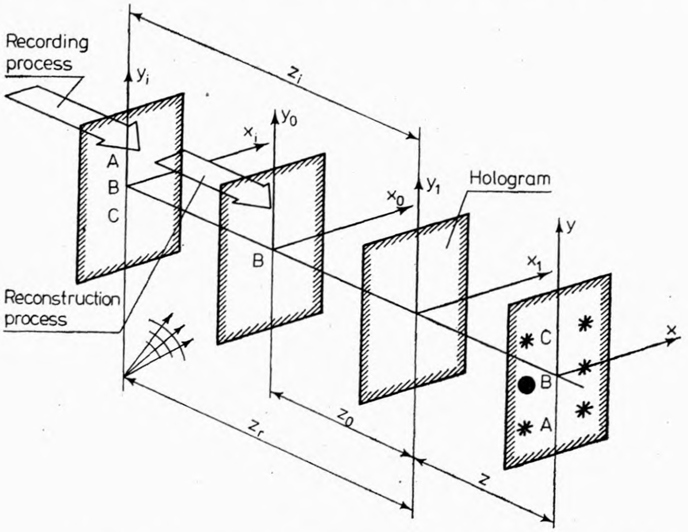

Fig. 4. Geometry of generalized Vander-Lugt filter

image plane:

$$W(x, y) = \iiiint\limits_{-\infty}^{+\infty} \tilde{h}^*(\tilde{x}_0-\tilde{x}_i, \tilde{y}_0-\tilde{y}_i)\, U_0(x-\tilde{x}_0, y-\tilde{y}_0) \times$$
$$\times \hat{U}_i^*(\tilde{x}_i, \tilde{y}_i)\, d\tilde{x}_i d\tilde{y}_i d\tilde{x}_0 d\tilde{y}_0. \tag{16}$$

The Fourier transform of the above relation has the form:

$$\hat{W}(f_x, f_y) = B\,\hat{U}_0(f_x, f_y)\,\hat{U}_i^*(f_x, f_y) \times$$
$$\times P_{\text{eff}}(-\lambda z f_x, -\lambda z f_y;\ 0, 0)\exp(j2\pi s f_x z/z_r), \tag{17}$$

where:

$$\hat{W}(f_x, f_y) = \hat{F}\{W(x, y)\}, \tag{18}$$
$$\hat{U}_0(f_x, f_y) = \hat{F}\{U_0(\tilde{x}_0, \tilde{y}_0)\},$$
$$\hat{U}_i(f_x, f_y) = \hat{F}\{U_i(\tilde{x}_i, \tilde{y}_i)\}.$$

Thus the spatial filtering of the rescaled Fourier transform $\hat{U}_0$ and $\hat{U}_i^*$ takes place in the hologram plane.

Finally, it can be proved from eq. (17) that the system, considered above, performs the same task as Vander–Lugt filter. Moreover, it operates in more general geometry than the last one. In particular, when $P_{\text{eff}} = 1$, $m = m_0 = 1$, $s/z_r \to \Theta_r$ and $z_i \to F_i$ (the lens Fourier HOS), we obtain a conventional Vander–Lugt filter. Additionally, from eq. (17) it results that influence of the both OTF and geometry of the system on the resolving power of common (Fresnel) HOS is analogous to the case of generalized Vander–Lugt filter.

The geometry and OTF influence on the effective pupil function form is presented in fig. 5. The geometrical considerations, those illustrated in fig. 3, implicate the following relation for rectangular hologram plate with sizes L_x, L_y:

$$\begin{aligned} P_{\text{eff}} &= P, \text{ for } b > 1, \\ P_{\text{eff}} &= P \cdot P_F, \text{ for } -1 \leqslant b \leqslant 1, \\ P_{\text{eff}} &= 0, \text{ for } b < 1, \end{aligned} \tag{19}$$

where the parameter b has the form:

$$b = \frac{2R_F}{L}\left[1 - \frac{(s/z_r)}{\lambda f_0}\right]. \tag{20}$$

From fig. 5 we can see that the interval $\Delta\Theta_r$, for $P_{\text{eff}} \neq P$ (OTF influence) increases with increasing $|z_i - z_r|$ and decreasing λf_0. For example, for $L_x = 10$ cm, $z_i = 20$ cm, $\lambda f_0 = 0.2$ and $z_r = 23$ cm, we get, from

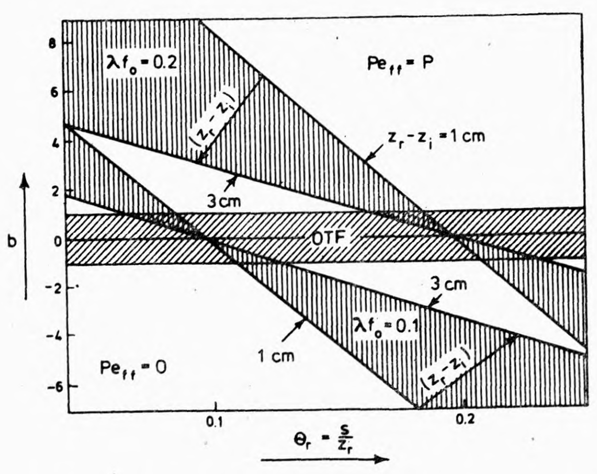

Fig. 5. Illustration of eqs. (19), (20), for $L_x = 10$ cm and $z_i = 20$ cm

fig. 5, $\Theta_r = 0.155$–0.24; hence $s \neq 3.65$–8.6 cm. Therefore, in practice the relation (17), with $P_{\text{eff}} \neq P$, is often of importance, even in conventional Vander–Lugt filter case.

Conclusions

Since the influence of the system geometry and OTF generate the phase--amplitude masks situated in hologram and object plane, the effective pupil formalism is very useful for system analysis of the HOS. Therefore, in terms of information theory, the HOS can be treated as a multi-channel optical filter with different two-dimensional spatial frequency pass-bands for several isoplanatic regions of the object plane. Next, by treating the HOS, as a subsystem of a more general system, the analysis of the latter can be carried out, as for instance, for the holographic system called in this paper generalized Vander–Lugt filter.

The further generalization of the system, by adding the third object situated in plane z_r = constant, does not introduced any significant complications (such a system is connected with Gabor's idea of associative memories [9]). Also, the influence of the additional effects as: partial coherence [10] and Gaussian shape of the beams can be easily adapted.

Finally, the above procedure allows a global analysis of the linear holographic system in terms of structural information formalism.

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Обобщенный фильтр Вандер — Люгта

Применён формализм эффективного зрачка для анализа модифицированной голографической системы с более общей геометрией, чем фильтр Вандер–Люгта. Обсуждено влияние как геометрии системы, так и оптической функции переноса для плёнки на разрешающую силу.