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Fluctuations, oscillations and chaos in dispersive optical bistability*

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The appearance of oscillations and chaos in optical bistable devices with dispersive nonlinearity — mainly in the kind of THG — is shown. Further the influence of noise in the driving fields is taken into account in a suitable Fokker-Planck theory.

1. Introduction

The behaviour of optical bistable systems is strongly dependent on fluctuations both in the stationary states and in their time evolution, e.g., in switching processes. Therefore in any application random processes should be taken into account.

Fluctuations of the fields in the bistable devices may be generated by fluctuating driving fields or by the coupling to the nonlinear material. Besides chaotic behaviour may appear under full deterministic conditions in the nonlinear process, indeed on a quite other time scale.

In this paper both the possibilities are investigated. We consider dispersive optical bistability mainly with third order interaction in the kind of third harmonic generation. The model is defined in Section 2. In Section 3 it is treated in a pure deterministic fashion, where especially the appearance of self-pulsing and chaos will be shown. The assumption of driving field fluctuations in Section 4 leads to the Fokker-Planck equation for the probability distribution which can be solved for special cases.

2. The model

Let us regard a ring cavity filled with a third order nonlinear material. Inside the cavity two monochromatic modes are excited with angular frequencies ω_1, ω_2, where $\omega_2 = 3\omega_1$. The modes are driven externally. We assume exact phase matching and neglect propagation effects. The influence of the nonlinear material is taken into consideration by the frequency-depending third order

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susceptibility $\chi^{(3)}$. Assuming slow time variations of the amplitudes in the plane wave approximation, the Maxwell equations are reduced to differential equations of first order:

$$\begin{aligned} dX_1/d\tau &= Y_1 - X_1 - 3ia\ X_1^{*2}X_2, \\ dX_2/d\tau &= a^2(Y_2 - X_2 - iaX_1^3), \\ a &= (\gamma_2/\gamma_1)^{1/2},\ \tau = \gamma_1 t. \end{aligned} \tag{1}$$

The dimensionless complex quantities X_μ, Y_μ are proportional to the resonator mode amplitudes E_μ and to the driving field amplitudes $E_\mu^{(d)}$, respectively. We have

$$X_\mu = -i(\gamma_\mu K)^{1/2}E_\mu,\ Y_\mu = -i(KT/\gamma_\mu)^{1/2}(c/L)E_\mu^{(d)}, \tag{2}$$

with

$$K = (2\pi\omega_1\chi^{(3)}/\gamma_1\gamma_2\varepsilon_1)[(2\varepsilon_1/\varepsilon_2)^{(1-2\delta_{2u})}]^{1/2}$$

where L means the effective resonator length, T — the transmittivity of two resonator mirrors, ε_μ — the permittivity. The real parts of the quantities γ_μ describe damping by absorption and transmission, the imaginary parts express the detuning between the cavity resonances and the driving field frequencies.

With $X_\mu = x_\mu e^{i\varphi_\mu}$, $Y_\mu = y_\mu e^{i\psi_\mu}$ and if we choose without loss of generality $\psi_2 = 0$, we get

$$\begin{aligned} dx_1/d\tau &= y_1\ \cos(\psi_1-\varphi_1) - x_1 + 3ax_1^2x_2\ \sin(\varphi_2 - 3\varphi_1), \\ d\varphi_1/d &= (1/x_1)[y_1\sin(\psi_1-\varphi_1) - 3ax_1^2x_2\cos(\varphi_2-3\varphi_1)], \\ dx_2/d\tau &= a^2[y_2\cos\varphi_2 - x_2 - ax_1^3\sin(\varphi_2-3\varphi_1)], \\ d\varphi_2/d &= (a^2/x_2)[-y_2\sin\varphi_2 - ax_1^3\cos(\varphi_2-3\varphi_1)]. \end{aligned} \tag{3}$$

3. Deterministic treatment

3.1. Steady states

Setting $dX_\mu/d\tau = 0$ ($\mu = 1, 2$) we find the steady states of the system. Especially, for $\psi_1 = -\pi/6$ we get for any value $y_1 \geqslant 0$

$$y_1 = x_1^{(0)}[3ax_1^{(0)}(ax_1^{(0)3} - y_2) + 1] \tag{4}$$

(the upper index 0 indicates the steady state). If y_2 satisfies

$$\frac{3}{2}\sqrt{\frac{\sqrt{5}}{3a}} < y_2 < \frac{4}{3}\sqrt{\frac{1}{3a}}, \tag{5}$$

there exists a certain interval of values y_1 for which the function $x_1^{(0)} = f(y_1)$ is multivalued. For any value of y_1 inside this interval there are three values of $x_1^{(0)}$ satisfying Eq. (4). For y_1 outside the interval the function $x_1^{(0)} = f(y_1)$ will be singlevalued (Fig. 1). With the usual linear stability analysis we find the following condition for asymptotic stability

$$y_2 < \frac{2\sqrt[4]{2(1+a^2)^3}}{3\sqrt{3a}}, \text{ and } dy_1/dx_1^{(0)} > 0. \tag{6}$$

This condition is fulfilled for only two of the stationary solutions $x_1^{(0)}$ inside the interval, mentioned above: we have optical bistability.

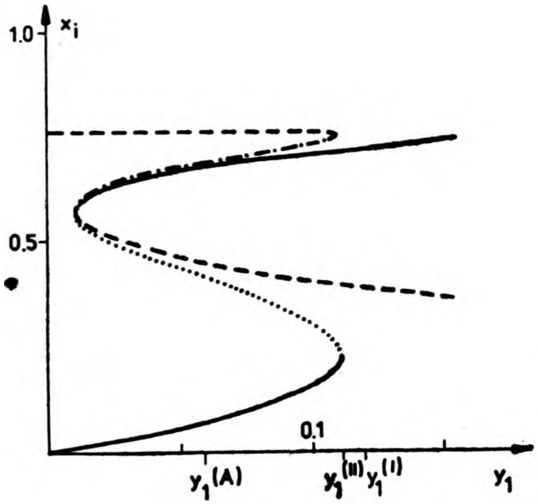

Fig. 1. State equations: mode amplitudes x_1 (————), and x_2 (– – – –) vs. driving field y_1 (Eq. (4)) for $y_2 = 0.76$ (...... and —·—·—·— unstable branches)

3.2. Time evolution of states

Switching processes are simulated by numerical solutions of the system of differential Eqs. (3). Figures 2 and 3 show the time evolution of the amplitudes of modes 1 and 2 for the case when, at times $\tau < 0$, the system was in a steady state on the lower branch in the midst of the bistable range. Then at $\tau = 0$ the driving field y_1 of the mode 1 is suddenly increased. After a certain time the system reaches the new stable state at the upper branch. By comparison of Figs. 2 and 3 one can see the effect of critical slowing down. The nearer the system comes to the marginal stability point ($dy_1/dx_1^{(0)} = 0$) the slower is its time evolution.

3.3. Self pulsing and chaos

We consider the case of generation of the third harmonic ($y_2 = 0$). In view of Eq. (5) there is no bistable region. Assuming $\varphi_1(t) - \psi_1 = \pi/6$ we get from Eq. (1) $x_2^{(0)} = ax_1^{(0)3}$. Investigating the stability of the steady states by linear stability analysis we find a critical value

$$y_1^{\mathrm{cr}} = [(1+a^2)/6a^2]^{1/4}(3+a^2)/2. \tag{7}$$

For $y_1 < y_1^{cr}$ all the roots of the characteristic equation have negative real parts. The corresponding stationary solutions of the Eqs. (3) are stable. For $y_1 = y_1^{cr}$ the real parts of two of the four roots vanish

$$\lambda_{3/4} = 0 \pm ia\sqrt{(3+a^2)/2}\,. \tag{8}$$

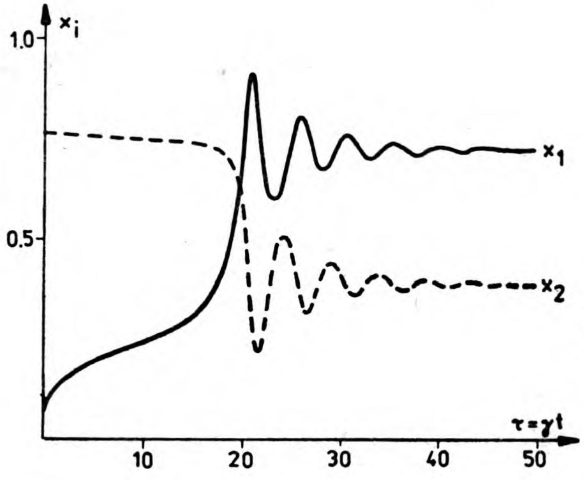

Fig. 2. Time evolution: mode amplitudes $x_i (i = 1, 2)$ for the case $y_1^{(A)} \to y_1^{(I)} = 0.12$ at $\tau = 0$ ($\gamma_1 = \gamma_2 = \gamma$, $y_2 = 0.76$, $\varphi_1(\tau) = \psi_1 = -\pi/6$, $\varphi_2(\tau) = \psi_2 = 0$)

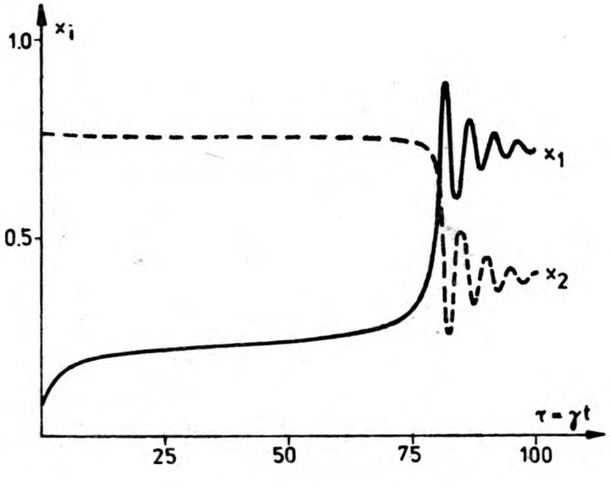

Fig. 3. Time evolution as in Fig. 2, but with $y_1^{(A)} \to y_1^{(II)} = 0.112$ at $\tau = 0$

If y_1 increases the real parts of $\lambda_{3,4}$ become positive. This means that a Hopf bifurcation takes place for $y_1 = y_1^{cr}$. By virtue of the Hopf bifurcation theorem periodic solutions in the form of stable limit cycles are expected in the neighbourhood of this point. Such a solution is given for $y_1 = 2$ in Fig. 4. Our numerical calculations show further bifurcations to period doubling with the increasing y_1 (Fig. 5). In the limit we get a chaotic behaviour (Fig. 6), even under full deterministic treatment.

For a similar device with trilinear (second order) interaction in the nonlinear material [1] Hopf bifurcations may appear too. Under suitable conditions we find critical points of the same character as those given by Eq. (8). An example of phase space trajectories can be seen in Fig. 7.

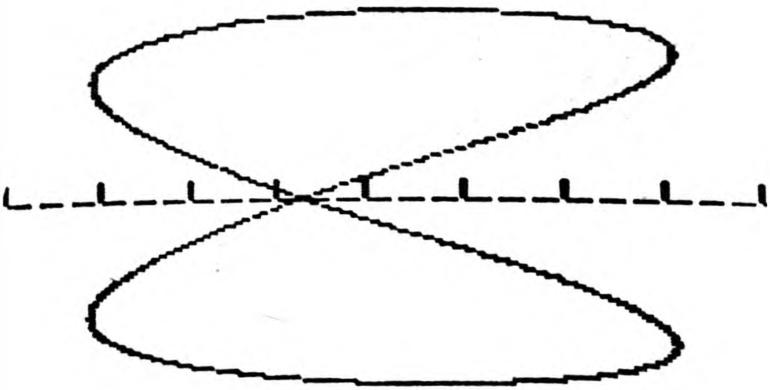

Fig. 4. Oscillation: trajectory in the complex plane of X_1 for $y_1 = 2$

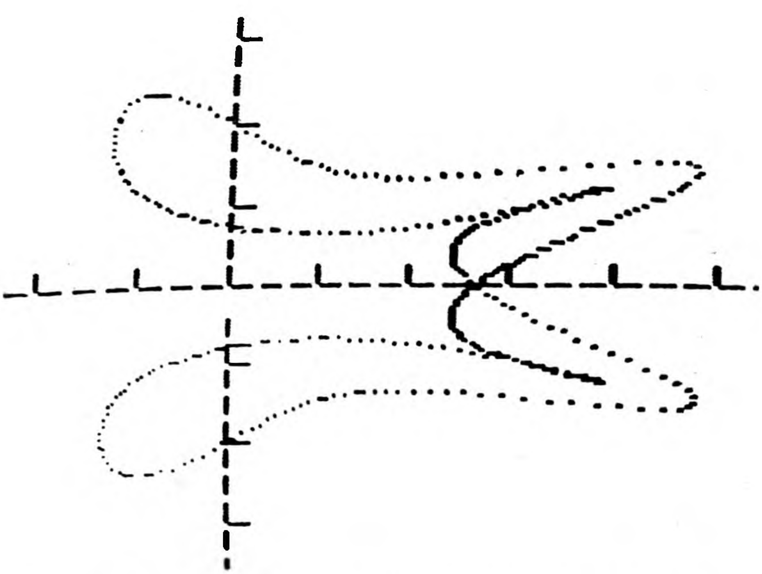

Fig. 5. Oscillation: trajectory in the complex plane of X_1 for $y_1 = 10$

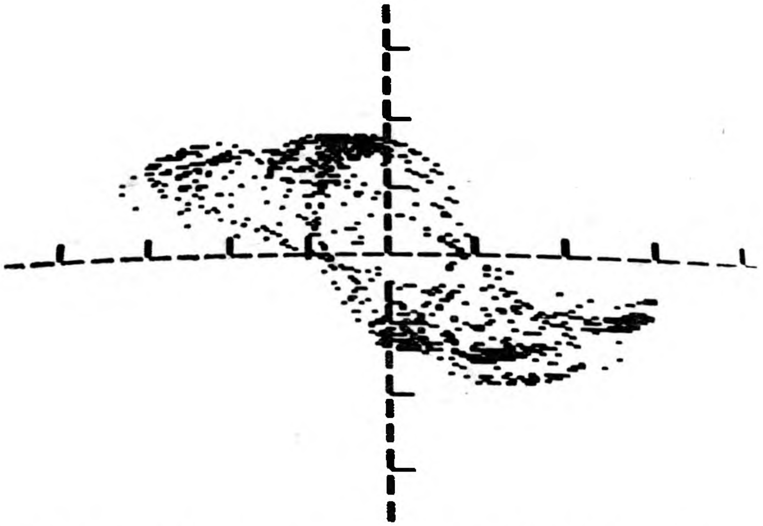

Fig. 6. Chaotic behaviour: complex plane of X_1 for $y_1 = 60$

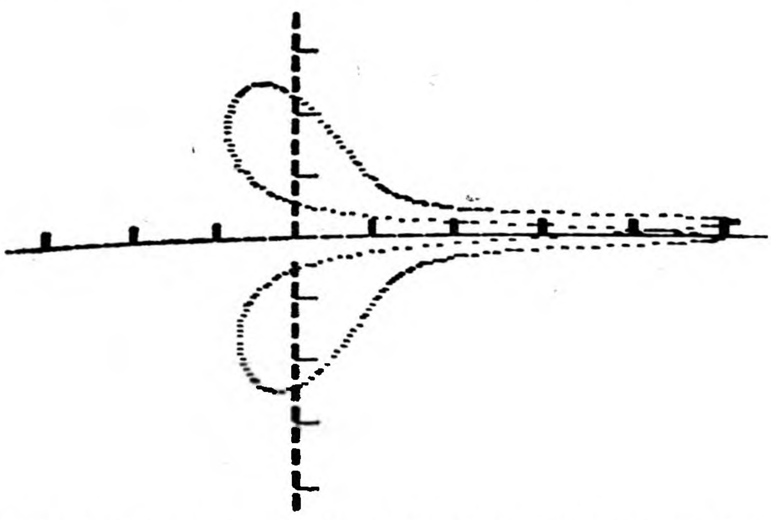

Fig. 7. Oscillations under trilinear interaction: trajectories in the complex plane of X_1 (one of the two ground modes) for $y_1 = y_2 = 60$

4. Driving fields with fluctuations

4.1. Langevin equations

Now we consider Y_μ as stochastic processes. We assume

$$Y_\mu(\tau) = Y_{\mu 0} + Y_{\mu 1}(\tau), \quad \mu = 1, 2 \tag{9}$$

where $Y_{\mu 1}(\tau)$ are assumed to behave as Gaussian white noise with

$$\langle Y_{\mu 1}(\tau) \rangle = 0, \tag{10}$$

$$\langle Y_{\mu 1}(\tau) Y_{\nu 1}(\tau')\rangle = \delta_{\mu\nu} D_\nu \delta(\tau-\tau'), \tag{11}$$

$$\langle Y_{\mu 1}(\tau) Y_{\nu 1}(\tau')\rangle = \langle Y^*_{\mu 1}(\tau) Y^*_{\nu 1}(\tau')\rangle = 0. \tag{12}$$

Most probably white noise does not exist in nature, whereas $Y_{\mu 1}$ should be the processes with finite correlation times τ_c (coloured noise). We assume, however, that the relaxation times of our system are much greater than τ_c. Then in a first approximation we may consider the limit $\tau_c \to 0$ (white noise).

We define

$$\xi_\mu = Y_{\mu 1}(\tau)/\sqrt{D_\mu}. \tag{13}$$

With Eqs. (9) and (13) we get from the Eq. (1) a set of nonlinear Langevin equations

$$\begin{aligned} dX_1/d\tau &= f_1 + \sqrt{D_1}\,\xi_1, \\ dX_2/d\tau &= f_2 + a^2\sqrt{D_2}\,\xi_2 \end{aligned} \tag{14}$$

where

$$\begin{aligned} f_1 &= Y_{10} - X_1 - 3iaX_1^{*2}X_2, \\ f_2 &= a^2(Y_{20} - X_2 - iaX_1^3). \end{aligned} \tag{15}$$

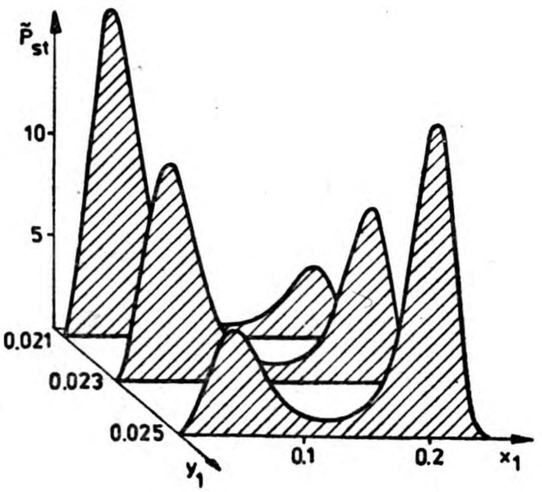

Fig. 8. $\tilde{P}_{st}(x_1)$ for several values of y_1 and for $y_2 = 0.23$, $\gamma_2/\gamma_1 = 100$, $D_1 = 5\times 10^{-4}$

The stochastic forces in the Langevin Eqs. (14) are multiplied by factors independent of X_μ. Therefore (X_1, X_2) is an additive stochastic process [2]. (X_1, X_2) is uniquely defined by Eqs. (14), if fixed initial values are given. No additional interpretation rules as in multiplicative processes are required [3].

4.2. Fokker-Planck equation

From the Eqs. (14) and using the Îto calculus we get the Fokker-Planck equation for the probability density $P(X_1, X_1^*, X_2, X_2^*, \tau)$

$$\frac{\partial P}{\partial \tau} = -\frac{\partial J_1}{\partial X_1} - \frac{\partial J_2}{\partial X_2} + \text{c.c.} \tag{16}$$

where

$$J_1 = \left(f_1 - D_1 \frac{\partial}{\partial X_1^*}\right) P,$$
$$J_2 = \left(f_2 - a^2 D_2 \frac{\partial}{\partial X_2^*}\right) P. \tag{17}$$

While trying to find a stationary solution of Eq. (16) we are confronted with a further difficulty: the potential conditions are not fulfilled. So we were able to solve the problem only under further specialisations.

We assume that mode 2 is driven by a stabilized laser ($D_2 = 0$) and is much more damped than the mode 1 ($\gamma_1 \ll \gamma_2$). The last assumption means that the mode 1 trails the mode 2, which may be eliminated adiabatically. In doing so the Eq. (16) is simplified to

$$\frac{\partial P(X_1, X_1^*, \tau)}{\partial \tau} = -\frac{\partial J_1}{\partial X_1} - \frac{\partial J_1^*}{\partial X_1^*} \tag{18}$$

where

$$J_1 = \left(\tilde{f}_1 - D_1 \frac{\partial}{\partial X_1^*}\right) P(X_1, X_1^*, \tau), \tag{19}$$

$$\tilde{f}_1 = Y_{10} - X_1(1 + 3a^2 |X_1|^4) - 3iaY_{20}X_1^{*2}. \tag{20}$$

We find a steady state solution of Eq. (18) with $I_1 = 0$ (potential case):

$$P_{st}(X_1, X_1^*) = N \exp[-U(X_1, X_1^*)/D_1] \tag{21}$$

where

$$U(X_1, X_1^*) = -Y_{10}X_1^* - Y_{10}^*X_1 + |X_1|^2(1 + a^2|X_1|^4) - ia(Y_{20}^*X_1^3 - Y_{20}X_1^{*3}). \tag{22}$$

In polar coordinates and if, without loss of generality, we choose $\psi_2 = 0$ the Eq. (22) reads

$$U(x_1, \varphi_1) = x_1^2(1 + a^2x_1^4) - 2y_{10}x_1 \cos(\psi_{10} - \varphi_1) - 2ay_{20}x_1^3 \cos[3(\varphi_1 + \pi/6)]. \tag{23}$$

So we may write

$$P_{st}(x_1, \varphi_1) = N \exp\left\{-\frac{1}{D_1} [x_1^2(1 + a^2x_1^4) - 2y_{10}x_1 - 2ay_{20}x_1^3]\right\}$$
$$\times \exp\left\{-\frac{\sin^2[(\psi_{10} - \varphi_1)/2]}{D_1/4y_{10}x_1}\right\} \exp\left\{-\frac{\sin^2[3(\varphi_1 + \pi/6)/2]}{D_1/4y_{20}ax_1^3}\right\}. \tag{24}$$

In the special case $\psi_{10} = -\pi/6$ the extrema of $P_{st}(x_1, \varphi_1)$ coincide with those of the state equation (4). In the same case the last two factors of Eq. (24) are Gaussians in $\sin(\varphi_1 + \pi/6)/2$ and $\sin 3(\varphi_1 + \pi/6)/2$, respectively. They are centred at $\varphi_1 = -\pi/6$ and have the variances $D_1/8y_{10}x_1$ and $D_1/8ay_{20}x_1^3$, respectively.

For a rough information we factorize $\tilde{P}_{st}(x_1, \varphi_1) = x_1 P_{st}(x_1, \varphi_1)$ and approximate the φ_1 — dependent part by a δ-function. So we get the functions $\tilde{P}_{st}(x_1)$ plotted in Fig. 8.

References

[1] Schütte F.-J., Germey K., Tiebel R., Worlitzer K., Optica Acta **30** (1983), 465.
[2] Schenzle R., Brand H., Phys. Rev. **20A** (1979), 1628.
[3] Van Kampen N. G., J. Stat. Phys. **24** (1981), 175.

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Флуктуации, осцилляции и хаос в прелелах оптической дисперсионной бистабильности

Показано появление осцилляции и хаоса в оптических бистабильных приборах с дисперсионной нелинейностью – прежде всего типа THG. Учитывается влияние шума в управляющем поле на основе соответствующей теории Фоккера-Планка.