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Letters to the Editor

Hartley spectrum of complex objects

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In this letter, the modulus and phase of the Hartley spectrum of selected phase and amplitude-phase objects evaluated numerically are presented and compared with those of the Fourier spectrum. The optical relations between the Fourier, Hartley and Hilbert spectra are shown.

1. Introduction

MAGIERA and MAGIERA in [1] showed some properties of the Fourier transform (FT) spectrum, pointing out in particular that the FT spectrum is usually complex even if the object is real. The phase of the FT spectrum contains essential information, while the available detectors react only to insensitive being intensive to phase. Therefore, the loss of phase of the FT spectrum in the optical system may be a serious defect of the latter. The Hilbert transform (Hi), as it was shown by MAGIERA and PLUTA [2], is equivalent to filtration during which the amplitudes of the spectrum components remain unchanged, while the phase is shifted by $\pi/2$ in either positive or negative direction. In paper [3], GAJ, MAGIERA and PLUTA showed the optical analog methods of realization of the FT and Hi spectra. This analog system in the two-dimensional version presented in [3] has been repeated for convenience as Fig. 1 in this letter. The Hartley spectrum, as it was shown by DONG, GU, YANG [4], [5], is real and the HT phase of the real object contains only binary values 0 or π.

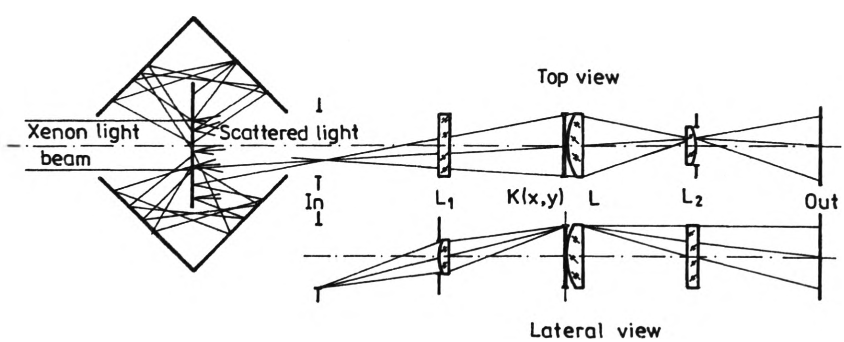

Fig. 1. Schematic representation of a non-coherent realizing an integral information. L_1 and L_2 – cylindrical lenses, $K(x,y)$ – filter realizing transform kernel, L – spherical lens

In this work, the modulus and phase of the HT spectra of selected one-dimensional complex objects have been examined and compared with the FT spectrum. In the further part of this paper, the relations between the FT, HT and Hi have been determined.

2. Theory and numerical results

The Hartley transform is of real kernel. For one-dimensional complex object $t(x) = |t(x)|e^{i\Phi(x)}$, the real transform Hartley $H(u)$ is defined as follows [4]:

$$H(u) = \int_{-\infty}^{\infty} t(x)[\cos(2\pi ux)+\sin(2\pi ux)]dx. \tag{1}$$

The inverse Hartley transform is of the form

$$t(x) = \int_{-\infty}^{\infty} H(u)[\cos(2\pi ux)+\sin(2\pi ux)]du. \tag{2}$$

The Hartley spectrum of a complex object may be determined when are known its real $H_{Re}(x)$ and imaginary $H_{Im}(x)$ parts:

$$|H(x)| = \sqrt{H_{Re}^2(u)+H_{Im}^2(x)}\ , \tag{3a}$$

and the phase is given by

$$\Phi_H(x) = \arctan\frac{H_{Im}(u)}{H_{Re}(u)} \tag{3b}$$

where: $H_{Re}(x)$ – real part of HT spectrum

$$H_{Re}(x) = \int_{-\infty}^{\infty} |t(x)|\cos(\Phi(x))[\cos(2\pi ux)+\sin(2\pi ux)]dx, \tag{3c}$$

$H_{Im}(u)$ – complex part of HT spectrum

$$H_{Im}(x) = \int_{-\infty}^{\infty} |t(x)|\sin(\Phi(x))[\cos(2\pi ux)+\sin(2\pi ux)]\mathrm{d}x. \tag{3d}$$

The modulus $|H(u)|$ and phase $\Phi_H(u)$ were determined for the following phase and amplitude-phase objects given by the following formulas:

i) $t(x) = 1.8\ \exp\{i[\sin(\pi/1.2(x-0.4))]\}$, $|x-0.4| \leqslant 0.6$.

Here, the obtained results are presented in Figs. **2a**,**b** and **3a**,**b**. The modulus and phase of the FT corresponding spectrum are shown in Figs. **4a**,**b**.

ii) $t(x) = 1.8\ \sin(\pi/1.2(x-0.4))\exp\{-1.5i(x+0.6)\}$, $|x-0.4| \leqslant 0.6$.

The respective modulus and phase of the HT are shown in Figs. **5a**,**b** and the corresponding FT spectrum in Figs. **6a**,**b**.

The objects i) and ii) were selected so that their moduli $|F(u)|$ were different and their phases $\Phi_F(u)$ – the same (Figs. **4a**, **6a** and **4b**, **6b**), while their $|H(u)|$ were similar and $\Phi_H(u)$ different (Figs. **2a**, **5a** and **2b**, **5b**).

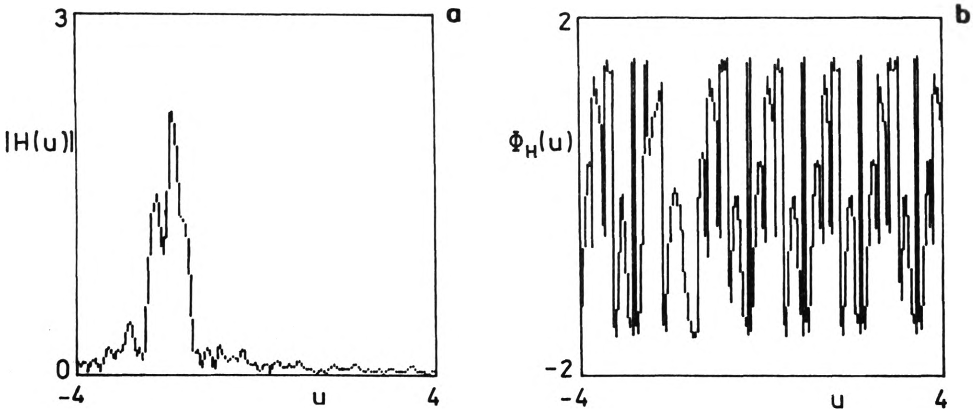

Fig. 2. The Hartley spectrum of $t(x) = 1.8\exp\{i[\sin(\pi/1.2(x-0.4))]\}$, $|x-0.4| \leqslant 0.6$. a — modulus $|H(u)|$, b — phase $\Phi_H(u)$

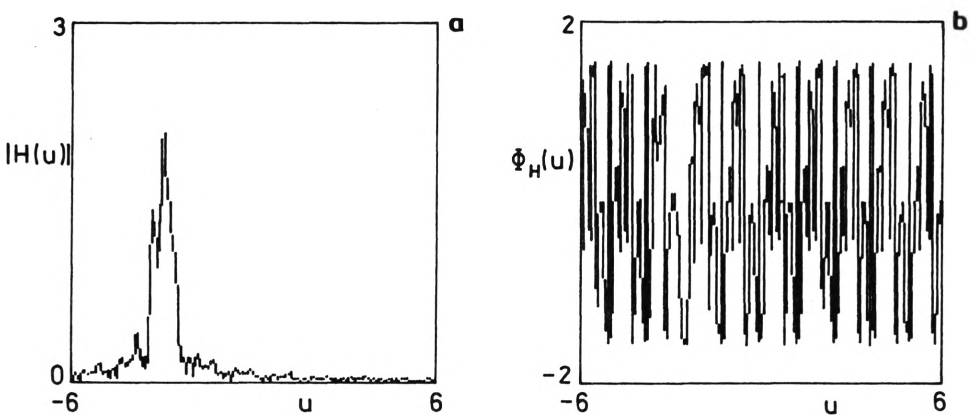

Fig. 3. The same, as in Fig. 2, but within the interval $[-6, 6]$

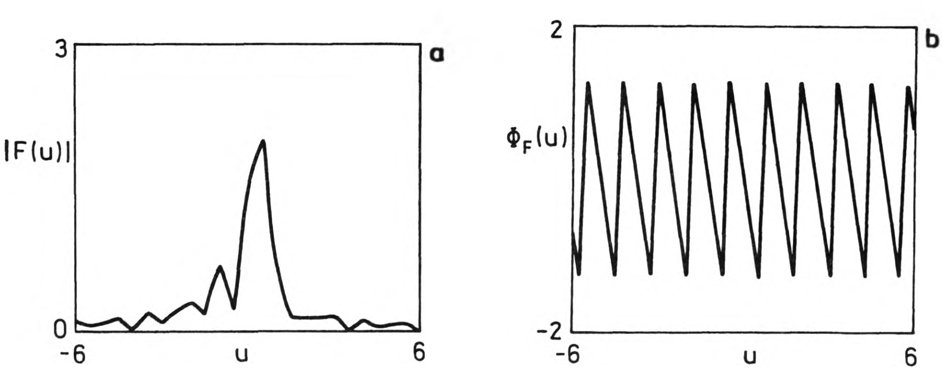

Fig. 4. Fourier spectrum of $t(x) = 1.8\exp\{i[\sin(\pi/1.2(x-0.4))]\}$, $|x-0.4| \leqslant 0.6$. a — modulus $|F(u)|$, b — phase $\Phi_F(u)$

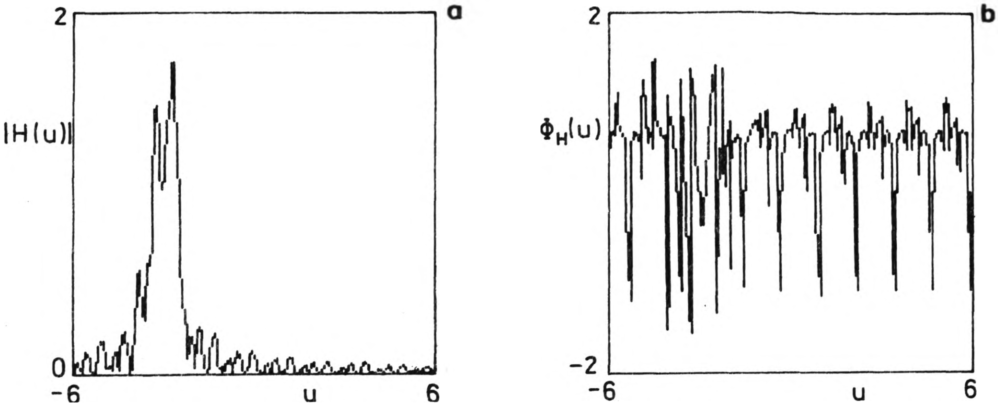

Fig. 5. The HT spectrum of $t(x) = 1.8\,\sin(\pi/1.2(x-0.4))\exp\{-i1.5(x+0.6)\}$, $|x-0.4| \leqslant 0.6$. a — modulus $|H(u)|$, b — phase $\Phi_H(u)$

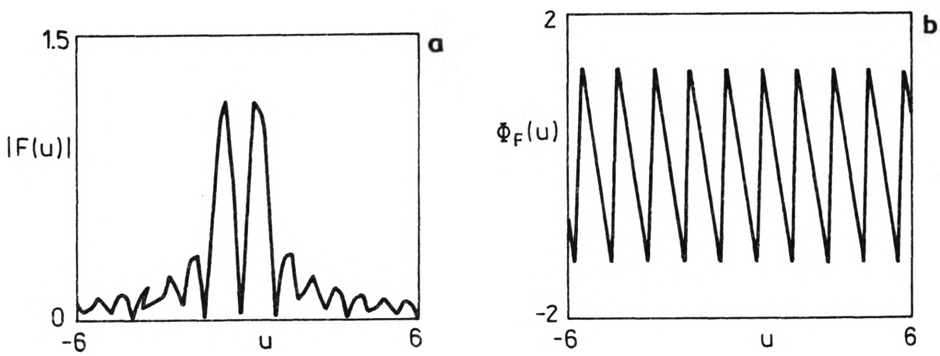

Fig. 6. The FT spectrum of $t(x) = 1.8\sin(\pi/1.2(x-0.4))\exp\{i1.5(x+0.6)\}$, $|x-0.4| \leqslant 0.6$. a — modulus $|F(u)|$, b — phase $\Phi_F(u)$

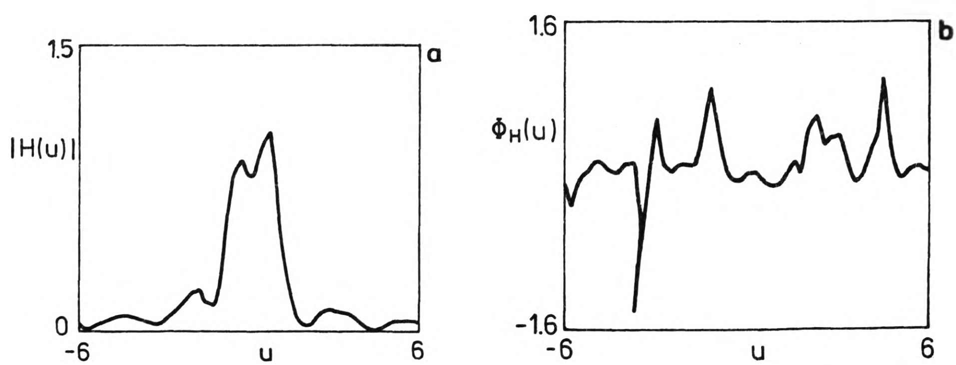

Fig. 7. The Hartley spectrum of $t(x) = \dfrac{2\sin(2\pi x)}{(2\pi x)} + i\left(\dfrac{\sin(2\pi x)}{(-2\pi)} - \dfrac{\sin(6\pi x)}{(6\pi)}\right)$. a — $|H(u)|$, b — $\Phi_H(u)$

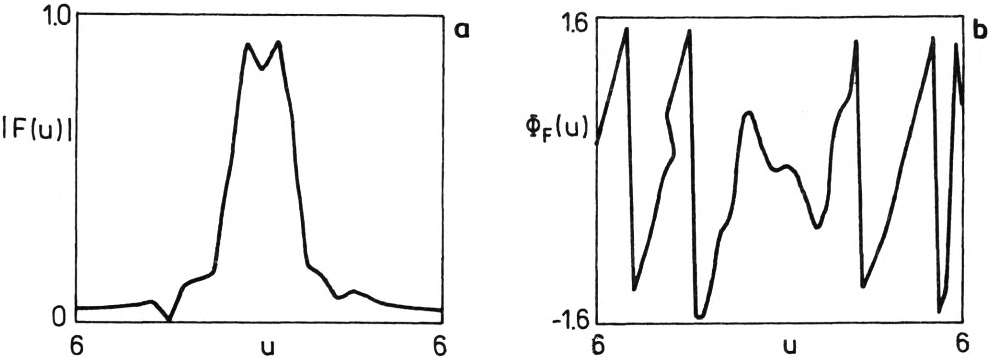

Fig. 8. The Fourier spectrum of $t(x) = \frac{2\sin(2\pi x)}{(2\pi x)} + i\left(\frac{\sin(2\pi x)}{(-2\pi)} - \frac{\sin(6\pi x)}{(6\pi)}\right)$. **a** — modulus $|F(u)|$, **b** — phase $\Phi_F(u)$

iii) $t(x) = 2\frac{\sin(2\pi x)}{(2\pi x)} + i\left[\frac{\sin(2\pi x)}{(-2\pi)} - \frac{\sin(6\pi x)}{(6\pi)}\right], \quad |x-0.4| \leqslant 0.6.$

The HT spectrum of this object is shown in Fig. 7**a**,**b** and its Fourier spectrum is presented in Fig. 8**a**,**b**. Here the object was chosen in this form to evaluate the differences between $|F(u)|$ and $|H(u)|$ as well as between $\Phi_F(u)$ and $\Phi_H(u)$, (Figs. 7**a**, 8**a** and 7**b**, 8**b**). From the results obtained, it follows that the phase of the HT spectrum of the complex objects is not of the binary form $[0, \pi]$ as it is the case for real objects (see [4]), but it takes the form of a continuous function of values contained between 0 and π (Figs. 4**b**, 6**b** and 7**b**, 8**b**).

3. Application of the Hartley transform

The Hartley transform can be applied in order to obtain the Hilbert transform, which is useful to examine the phase shown in paper [2]. The HT spectrum may be expressed as a combination of the complex Fourier transform and the real and imaginary parts of the latter [4]

$$H(u) = 1/2(1+i)[F(u)+iF^*(u)] - F_{Re}(u) - F_{Im}(u), \tag{4}$$

where: $F_{Re}(u)$ — real part of the FT spectrum, $F_{Im}(u)$ — imaginary part of the FT spectrum.

$F_{Re}(u)$ and $F_{Im}(u)$ may by expressed by the Hilbert transform [2] as follows:

$$F_{Re}(u) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F_{Im}(x')}{(x'-x)}dx',$$

$$F_{Im}(u) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F_{Re}(x')}{(x'-x)}dx'. \tag{5}$$

When performing twice the Hartley transform of an object function, the Hilbert transform of this function is obtained [2]

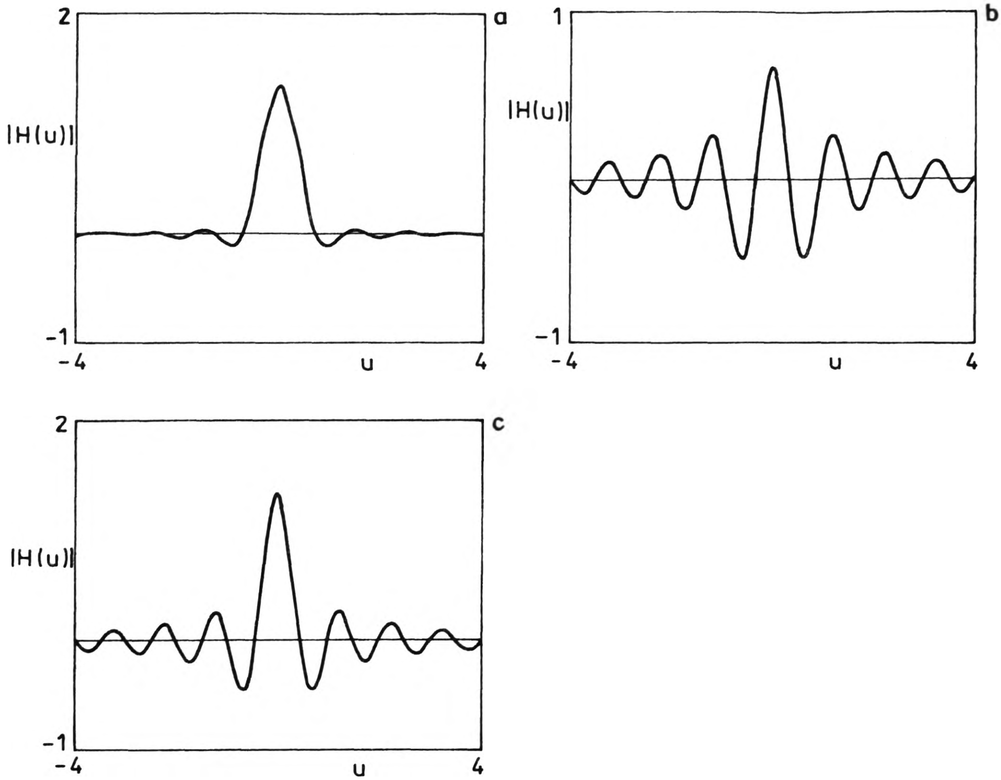

Fig. 9. The HT spectrum of: a — $t(x) = 1-x^2$, b — $t(x) = x^2$, c — $t(x) = 1/2(1+x^2)$

$$\mathscr{F}_{\rm H}\{\mathscr{F}_{\rm H}\{t(x)\}\} = \mathscr{F}_{\rm HI}\{t(x)\}, \tag{6}$$

which was shown below for the aperture apodized with a filter.

Let the object being the subject of the analog optical imaging be an aperture apodized with the transmission of the form

$$\text{i) } t(x) = \begin{cases} 1-x^2 & |x| \leqslant a \\ 0 \end{cases}$$

considered previously in the paper [2].

The HT spectrum of this function is equal to

$$H(u) = \left(\frac{1}{\pi u} - \frac{a^2}{\pi u} + \frac{1}{2\pi^3 u^3} - \frac{a}{\pi^2 u^2}\right)\sin(2\pi ua)$$

and corresponds to the FT spectrum of $A(x)$, *i.e.*, amplitude point spread function, where:

$$A(x) = \mathscr{F}_{\rm F}\{1-x^2\}.$$

$H(u)$ is shown in Fig. 9c.

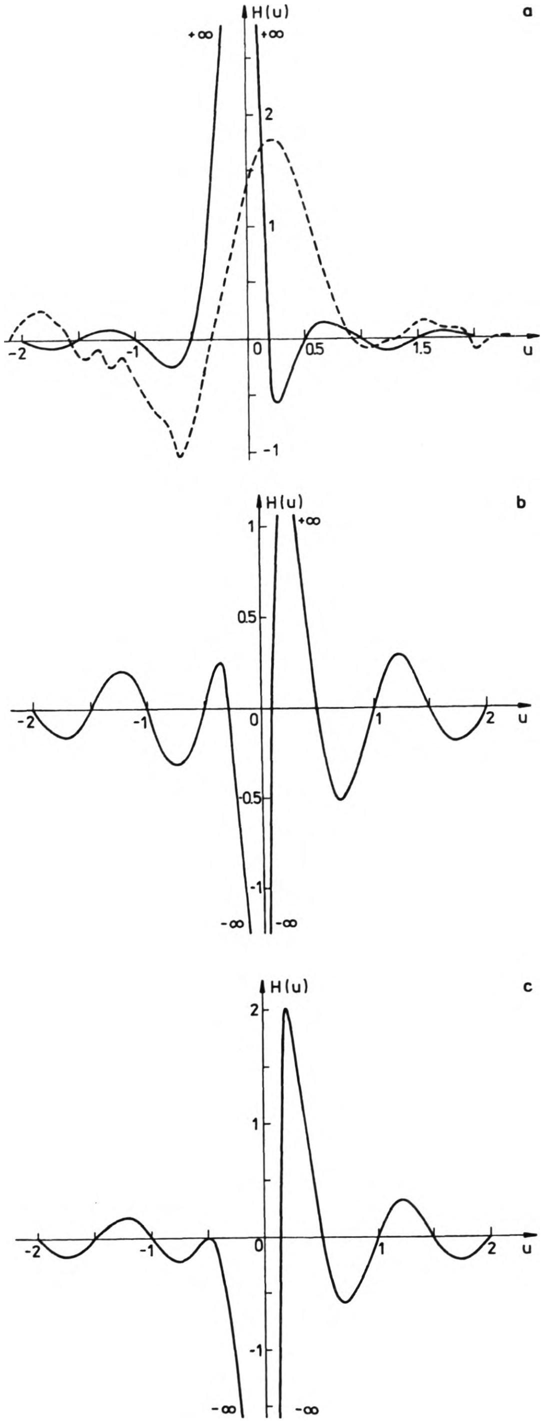

Fig. 10. The Hilbert spectrum of: a — $t(x) = 1 - x^2$ (——), $t(x) = 1.8\cos(\pi/1.2(x-0.4))$, $|x-0.4| \leqslant 0.6$ (− − −), b — $t(x) = 1/2(1+x^2)$, c — $t(x) = x^2$

Correspondingly for:

ii) $t(x) = \begin{cases} x^2 & |x| \leqslant a \\ 0, & \end{cases}$

$$H(u) = \left(\frac{a^2}{\pi u} - \frac{1}{2\pi^3 u^3} + \frac{a}{\pi^2 u^2}\right)\sin(2\pi ua) = \mathscr{F}_F\{1-x^2\}, \text{ (Fig. 9a).}$$

iii) $t(x) = \begin{cases} 1/2(1+x^2) & |x| \leqslant a \\ 0, & \end{cases}$

$$H(u) = (2\pi^2u^2 + 2a^2\pi^2u^2 - 1 + 2a\pi u)\frac{\sin(2\pi ua)}{4\pi^3 a^3} = \mathscr{F}_F\{1/2(1+x^2)\}, \text{ (Fig. 9b).}$$

The HT spectrum of the function $A(x)$ gives as a result the Hi spectrum shown in Fig. 10a,b,c, which means that

$$\mathscr{F}_H\{\mathscr{F}_H\{t(x)\}\} = \mathscr{F}_H\{\mathscr{F}_F\{t(x)\}\} = \mathscr{F}_{Hi}\{t(x)\}. \tag{7}$$

This relation may be realized in both coherent and incoherent two-channel analog processors proposed by MAGIERA and PLUTA in [2] or in a two-channel polarization system suggested by LI and EICHMANN in [6].

The Hartley transforms may be applied also to reconstructions of the objects, which for the case of amplitude objects was shown by DONG, GU and YANG in papers [4], [5]. The reconstruction of the phase and amplitude-phase objects will be the subject of further examination.

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References

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Errata

At the author's (A. Magiera) request, the following errata of her text published as a *Letter to the Editor* (Optica Applicata, Vol. 24 (1994), No. 3, entitled *Hartley spectrum of complex objects*, is printed below:

In caption of Fig. 6 (p. 182), instead of ... $\exp\{i1.5(x+0.6)\}$ ... .
should be ... $\exp\{-i1.5(x+0.6)\}$ ... ,

In page 184 (bottom line 1), is ... in Fig. 9c.
should be ... in Fig. 9a.

In page 186, instead of ... $= \mathcal{F}_{r}\{1-x^2\}$, (Fig. 9a).
should be ... $= \mathcal{F}_{r}\{x^2\}$, (Fig. 9b).

instead of ... , (Fig. 9b).
should be ... , (Fig. 9c).