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Paraxial rays in the geodesic lens

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Analytical solution of the ray trajectory equation in first-order approximation and results of numerical computations for two types of geodesic lenses have been presented.

1. Introduction

From the geometrical theory of geodesic lens it is known that this lens may be given as a surface $r(u, v)$ and a ray passing through the lens satisfies the differential equation (see [1], for example)

$$\frac{d^2 v}{du^2} = \begin{Bmatrix}1\\22\end{Bmatrix}\left(\frac{dv}{du}\right)^3 + \left[2\begin{Bmatrix}1\\12\end{Bmatrix} - \begin{Bmatrix}2\\22\end{Bmatrix}\right]\left(\frac{dv}{du}\right)^2 + \left[\begin{Bmatrix}1\\11\end{Bmatrix} - 2\begin{Bmatrix}2\\12\end{Bmatrix}\right]\frac{dv}{du} + \begin{Bmatrix}2\\11\end{Bmatrix} \tag{1}$$

where: u, v – some parameters,

$\begin{Bmatrix}k\\ij\end{Bmatrix}$ – Christoffel's symbols.

In general, this equation cannot be solved analytically.

Recently, various methods of numerical examination of ray tracing through geodesic lenses have been considered [2]–[5]. In his work [6] Sottini has given a general solution useful in the lens design as well as the analytical solution of the Abele's integral equation in a simple case.

In this paper, the method of analytical solution by paraxial approximation of the ray equation (1) has been proposed. This method bases on the means of solving of the ray equation in gradient index media in paraxial region [7].

2. Method presentation

Paraxial approximation of equation (1) is possible if we assume that parameter v is a sufficiently slow-changing function of parameter u. This condition is satisfied for $u = z$, $v = x$, where x, z are Cartesian coordinates on the waveguide plane (or, in general, on plane parallel to one), and z is the direction of light propagation. Then y is the axis of rotational symmetry of surface $r(x, z)$ (geodesic lens). That surface is determined by the generating curve $y = y(x^2+z^2)$. In this system of coordinates

Eq. (1) has the form

$$\frac{d^2 x}{dz^2} = \frac{y_z y_{xx}}{1+y_z^2+y_x^2}\left(\frac{dx}{dz}\right)^3 + \frac{2y_z y_{zx} - y_x y_{xx}}{1+y_z^2+y_x^2}\left(\frac{dx}{dz}\right)^2 + \frac{y_z y_{zz} - 2y_x y_{zx}}{1+y_z^2+y_x^2}\frac{dx}{dz} - \frac{y_x y_{zz}}{1+y_z^2+y_x^2} \tag{2}$$

where indices x, z denote partial derivatives of y in respect to x and z. If, in addition, we assume that incident rays pass near the z axis, then the solution of Eq. (2) may be given as the series

$$x = x_0 f^{(1)} + x_0^2 f^{(2)} + \ldots \tag{3}$$

where x_0 is x-coordinate of input point at the boundary line of the lens, and $f^{(1)}, f^{(2)}$ are functions of z which are to be determined.

After substituting (3) to (2) we obtain

$$x_0 \frac{d^2 f^{(1)}}{dz^2} + x_0^2 \frac{d^2 f^{(2)}}{dz^2} + \ldots = P(x, z)\left(x_0 \frac{df^{(1)}}{dz} + x_0^2 \frac{df^{(2)}}{dz} + \ldots\right)^3 + Q(x, z)\left(x_0 \frac{df^{(1)}}{dz} + x_0^2 \frac{df^{(2)}}{dz} + \ldots\right)^2 + R(x, z)\left(x_0 \frac{df^{(1)}}{dz} + x_0^2 \frac{df^{(2)}}{dz} + \ldots\right) + S(x, z) \tag{4}$$

where $P(x, z)$, $Q(x, z)$, $R(x, z)$ and $S(x, z)$ depend on the form of the generating curve as in (2). Consequently, to paraxial approximation neglecting in (4) terms with x_0 in power higher than first we obtain reduced differential equation

$$x_0 \frac{d^2 f^{(1)}}{dz^2} = \tilde{R}(x, z)\, x_0 \frac{df^{(1)}}{dz} + \tilde{S}(x, z) \tag{5}$$

where $\tilde{R}(x, z)$ and $\tilde{S}(x, z)$ are given by paraxial approximation of $R(x, z)$ and $S(x, z)$. This equation can be solved analytically for some forms of $y(x^2+z^2)$.

3. Examples of solutions

Let a spherical surface (spherical geodesic lens) be given

$$y = [r^2 - (x^2+z^2)]^{1/2}, \tag{6a}$$

and a paraboloidal surface

$$y = y_0 - \frac{1}{2} a (x^2 + z^2) \tag{6b}$$

where r, y_0 and a are constant parameters.

For the case (6a) Eq. (5) has the form

$$\frac{d^2 f^{(1)}}{dz^2} = \frac{z}{r^2 - z^2}\frac{df^{(1)}}{dz} - \frac{1}{r^2 - z^2} f^{(1)}; \tag{7a}$$

and for (6b)

$$\frac{d^2 f^{(1)}}{dz^2} = \frac{a^2 z}{1+a^2 z^2}\frac{df^{(1)}}{dz} - \frac{a^2}{1+a^2 z^2} f^{(1)}. \tag{7b}$$

Solutions of paraxial equations (7a) and (7b) are functions

$$f^{(1)} = -C_1 \frac{\sqrt{r^2 - z^2}}{r^2} + C_2 z, \tag{8a}$$

and

$$f^{(1)} \doteq C_1 [-\sqrt{1+a^2 z^2} + az \ln|az + \sqrt{1+a^2 z^2}|] + C_2 z, \tag{8b}$$

respectively.

Substituting (8a) and (8b) to (3) we finally obtain paraxial ray trajectories:

– for spherical lens (6a)

$$x = -x_0 C_1 \frac{\sqrt{r^2 - z^2}}{r^2} + x_0 C_2 z, \tag{9a}$$

– for lens with parabolic generating curve (6b)

$$x = x_0 C_1 [-\sqrt{1+a^2 z^2} + az \ln|az + \sqrt{1+a^2 z^2}|] + x_0 C_2 z. \tag{9b}$$

Integral constants C_1 and C_2 can be designed from boundary conditions as in paper [6].

4. Numerical results

Results of numerical computations have been presented in Tables 1 and 2 for spherical lenses, and in Tables 3 and 4 for paraboloidal lenses. Constant parameters y_0, a and r for two forms of lenses have been normalized with respect to the radius of lenses equal unity ($\sqrt{x^2 + z^2} = 1$) and $y = 0$ at the waveguide plane. In the Tables, both computations from paraxial formulae (9a), (9b) and results of numerical solutions of the Eq. (2) for examined lenses are confronted. Values x, y, z and dx/dz are shown for four points on trajectories of rays. Incident rays are parallel to the z-axis.

5. Conclusions

From the presented examples, we can conclude that the proposed method of solution of the ray trajectory equation (2) by its paraxial approximation leads, in some cases, to simple analytical formulae. Then, a ray trajectory can be examined without

Table 1. Spherical lens, $r = 1.512$

x_0		x		y		z		dx/dz		f	
		A	B	A	B	A	B	A	B	A	B
.1	1	.1	.1	0	0	−.9950	−.9950	.0331	.0331	2.006	2.001
	2	.1068	.1068	.2778	.2790	−.5304	−.5298	0	-5×10^{-5}		
	3	.1001	.1001	.3746	.3746	0	−.0025	−.0248	−.0247		
	4	.0504	.0503	0	0	.9987	.9987	−.0830	−.0831		
.2	1	.2	.2	0	0	−.9798	−.9798	.0645	.0645	2.037	2.026
	2	.2133	.2133	.2703	.2699	−.5180	−.5217	0	2×10^{-4}		
	3	.2004	.2004	.2646	.3646	0	−.0025	−.0483	−.0484		
	4	.1028	.1023	0	0	.9947	.9948	−.1642	−.1647		
.5	1	.5	.5	0	0	−.8660	−.8660	.1332	.1332	2.254	2.194
	2	.5265	.5261	.2153	.2143	−.4338	−.4395	0	4×10^{-4}		
	3	.5004	.5032	.2913	.2919	0	−.0022	.0999	−.1022		
	4	.2955	.2895	0	0	.95	.7572	−.3719	−.3791		
.7	1	.7	.7	0	0	−.7141	−.7141	.1432	.1432	2.603	2.507
	2	.7256	.7251	.1486	.1477	−.3384	−.3446	0	4×10^{-4}		
	3	.7071	.7057	.2024	.2032	0	−.0018	−.1074	−.1108		
	4	.4822	.4728	0	0	.8761	.8812	−.4400	−.4522		

Columns: A — paraxial computations from (9a), B — results of numerical solution of Eq. (2).
Lines: 1 — input point ($x = x_0$, $y = 0$), 2 — point $dx/dz = 0$, 3 — point $z = 0$, 4 — output point ($y = 0$).
f — focus distance (measured from centre of lens).

Table 2. Spherical lens, $r = 1.155$

x_0		x		y		z		dx/dz		f	
		A	B	A	B	A	B	A	B	A	B
.1	1	.1	.1	0	0	−.9950	−.9950	.0985	.0985	1.0100	1.0100
	2	.1155	.1153	.4209	.4190	−.5688	−.5746	0	2.7×10^{-4}		
	3	.1005	.1002	.5730	.5730	0	−.0025	−.0493	−.0494		
	4	.0010	.0003	0	0	1	.9999	−.2000	−.2007		
.2	1	.2	.2	0	0	−.9798	−.9798	.1884	.1884	1.0102	1.0385
	2	.2308	.2300	.4145	.4096	−.5443	−.5560	0	-2×10^{-4}		
	3	.2035	.2012	.5593	.5597	0	−.0024	−.0942	−.0966		
	4	.0076	.0021	0	0	.9999	.9999	−.3995	−.4058		
.5	1	.5	.5	0	0	−.8660	−.8660	.3464	.3464	1.0722	1.2055
	2	.5657	.5590	.3428	.3271	−.4082	−.4525	0	3.5×10^{-4}		
	3	.5292	.5126	.4490	.4576	0	−.0022	−.1732	−.1977		
	4	.0958	.0385	0	0	.9954	.9993	−.9526	−1.069		
.7	1	.7	.7	0	0	−.7141	−.7141	.3355	.3355	1.1830	1.3674
	2	.7633	.7572	.2380	2287	−.2930	−.3339	0	3×10^{-4}		
	3	.7383	.7233	.3104	.3230	0	−.0018	−.1678	−.2003		
	4	.2357	.1356	0	0	.9718	.9908	−1.164	−1.438		

Columns and lines identification as in Tab. 1.

Table 3. Paraboidal lens, $a = 0.4$, $y_0 = 2$

x_0		x A	x B	y A	y B	z A	z B	dx/dz A	dx/dz B	f A	f B
.1	1	.1	.1	0	0	−.9950	−.9950	.0077	.0077	5.349	5.352
	2	.1019	.1019	.1489	.1494	−.4949	−.4950	0	6×10^{-6}		
	3	.1000	.1000	.1980	.1980	0	−.0025	−.0079	−.0078		
	4	.0843	.0843	0	0	.9964	.9964	−.0234	−.0234		
.2	1	.2	.2	0	0	−.9798	−.9798	.0150	.0150	5.404	5.463
	2	.2037	.2037	.1439	.1447	−.4889	−.4874	0	3×10^{-5}		
	3	.1999	.2001	.1920	.1920	0	−.0025	−.0155	−.0153		
	4	.1693	.1696	0	0	.9856	.9853	−.0463	−.0460		
.5	1	.5	.5	0	0	−.8660	−.8660	.0327	.0327	5.900	6.360
	2	.5070	.5072	.1095	.1125	−.4422	−.4265	0	2×10^{-5}		
	3	.4992	.5003	.1502	.1499	0	−.0022	−.0351	−.0325		
	4	.4356	.4400	0	0	.9001	.8980	−.1056	−.1010		
.7	1	.7	.7	0	0	−.7141	−.7141	.0371	.0371	6.816	7.983
	2	.7063	.7068	.0722	.0761	−.3741	−.3481	0	1.2×10^{-4}		
	3	.6986	.7006	.1024	.1019	0	−.0018	−.0417	−.0360		
	4	.6331	.6422	0	0	.7741	.7665	−.1269	−.1156		

Columns and lines identification, as in Tab. 1.

Table 4. Paraboidal lens, $a = 1.0$, $y_0 = 0.5$

x_0		x A	x B	y A	y B	z A	z B	dx/dz A	dx/dz B	f A	f B
.1	1	.1	.1	0	0	−.9950	−.9950	.0410	.0410	1.0708	1.0786
	2	.1109	.1110	.3768	.3797	−.4838	−.4801	0	-8×10^{-5}		
	3	.0998	.1002	.4950	.4950	0	−.0025	−.0466	−.0457		
	4	.0066	.0073	0	0	1.0000	1.0000	−.1346	−.1347		
.2	1	.2	.2	0	0	−.9798	−.9798	.0798	.0798	1.0725	1.1049
	2	.2206	.2211	.3594	.3650	−.4822	−.4728	0	3×10^{-4}		
	3	.1987	.2010	.4803	.4798	0	−.0025	−.0925	−.0872		
	4	.0135	.0193	0	0	.9999	9998	−.2676	−.2678		
.5	1	.5	.5	0	0	−.8660	−.8660	.1625	.1625	1.0925	1.3136
	2	.5338	.5386	.2504	−.2756	−.4630	−.4005	0	2×10^{-5}		
	3	.4844	.5080	.3827	.3711	0	−.0022	−.2169	−.1563		
	4	.0417	.1327	0	0	.9991	.9912	−.6435	−.6153		
.7	1	.7	.7	0	0	−.7141	−.7141	.1721	.1721	1.1420	1.6524
	2	.7260	.7344	.1489	.1809	−.4185	−.3160	0	-7×10^{-4}		
	3	.6697	.7116	.2758	.2470	0	−.0018	−.2727	−.1456		
	4	.0875	.3315	0	0	.9962	.9434	−.8611	−.6958		

Columns and lines identification, as in Tab. 1.

time-consuming numerical computations of integrals. A convergence between paraxial trajectories and trajectories computed without approximation is sufficient for low values of x_0 and for lenses with long focus distance.

Paraxial optics of geodesic lenses does not have the means of a non-aberration optics, of course.

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Параксиальные лучи в геодезической линзе

В работе представлен метод аналитического решения уравнения луча в геодезической линзе в параксиальной апроксимации. Даны результаты нумерических вычислений траекторий лучей на основе полученных аналитических формул и они сравнены с траекториями лучей вычисленными без применения параксиальной апроксимации для двух типов геодезических линз.